Linear Water Wave Propagation around Structures

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Abstract: Objective of this contribution is to show how to implement the Mild Slope Equations with Comsol Multiphysics. These equations are commonly used to study the propagation of waves in harbors. Some interesting features are presented, namely the use of weak terms (used for the modelling of the source term); the evaluation of a smooth phase gradient from the complex dependent variable; a robust method to solve the wave dispersion relation with sufficient accuracy and continuity. As example application, the wave field that would occur in Casal Borsetti Marina (RA) accounting for a fictitious northward extension of the actual pears is shown.

Keywords: Mild Slope Equations, Wave reflection, Wave diffraction, Internal wave generation, Varying bed topography.

1. Introduction

Preliminary port and harbors design is frequently based on a numerical study of the wave propagation within the protected area. Significant excursions of the free surface in the harbour can be induced by external waves with period ranging from a few minutes to several hours. If the forcing lasts sufficiently long, the excited natural oscillations may cause failure of the mooring systems, fenders, berths, etc., resulting in heavy damage of to the boats and harbour structures.

Linear wave propagation in a gradually varying bottom and rapid horizontal variation due to reflection and diffraction is usually solved by means of the 2DH linear elliptic Mild Slope Equations (MSE) derived by Berkhoff (1972).

These equations are an enhancement of the Helmoltz Equations insofar as they account for the change of the wavenumber (*k*) with depth (*h*), usually in absence of currents. Of course the MSE degenerate into the Helmoltz Eq. for constant depth.

The term "mild" refers to an expansion in terms of a small parameter μ defined by (see for instance Mei, 1989):

$$\mu = O\left(\frac{\nabla h}{kh}\right) = 1 \qquad (1)$$

which relates the horizontal length scale of depth variation the wavelength. The solution of the MSE can be considered valid up to O(μ). Better approximations (i.e. to the order $O(\mu^2)$) of the MSE are given by Porter and Staziger (1995). Inclusion of currents, dissipations due to bottom friction and breakers can be accounted for as shown by Dingemans (1997). These common features will not be addressed here for the sake of simplicity.

For practical problems, where the bed topography is not simple, these equations are solved numerically. Finite Element Methods, which are typically associated to flexible meshes, appear suited to be applied in coastal areas, i.e. where part of the domain is very regular and can be described by large elements, whereas other parts usually need to be greatly refined.

Objective of this contribution is to show how to implement the MSE with Comsol Multiphysics in order to study the propagation of waves around structures.

In order to solve the MSE, the generic FE modeler shall face four difficulties (in ascending order):

1) solve the water wave dispersion, which is an implicit equation on the wave number;

2) define a complex dependent variable;

3) input waves with the method of the internal wave generation;

4) define boundary condition on absorbing or partially reflective structures.

Whereas the first problem is trivial for coastal engineers and the second is trivial for Comsol Multiphysics programmers, since there is no actual difference in the implementation of real or complex variables, the other problems may not be easy to solve without guidelines. For this purpose, the methods proposed by Bellotti et al. (2003) and Beltrami et al. (2001) are suggested, and details on their implementation are given.

This note is structured as follows: in Section 2 the boundary value problem is described and the method proposed by Bellotti et al. (2003) is set up in terms suited to the Comsol Multiphysics modeler. In Section 3 the model is validated against some benchmarks. Eventually, in Section 4, the fictitious harbour in Casal Borsetti (RA), as designed by students of the course of Harbour Construction, held by Prof. A. Lamberti, is analysed.

2. The Model

The MSE, given in Subsection 2.1, are elliptic and therefore appropriate values are required along the whole domain boundary. Unfortunately in some cases the boundary conditions also depend on the solution.

Parabolic approximations of the MSE are easier to solve with this regard, but they are not suited for application to harbour design. Indeed reflection is an essential feature to model in order to give a realistic response of the harbour to incident waves. Similarly, partial reflection, typically induced by rubble mound breakwaters, needs to be accounted for.

In Subsection 2.2, the simple method suggested by Beltrami et al. (2001) to account for partial reflection is presented. The method is based on an iterative approach, which is a standard way to add fancy features to the elliptic equations.

In Subsection 2.3 the method by Bellotti et al. (2003) is used to generate waves from the interior of the domain and avoid complications in the definition of the offshore boundary condition.

2.1 Governing Equations

The MSE are given, for instance, in Dingemans (1997).

Let the velocity potential be:

$$\Psi(x,t) = \mathrm{Re}[\psi(x)e^{i\omega t}] \tag{2}$$

where ω is the angular frequency of the wave, x is a 2D vector holding the horizontal spatial cohordinates. The dependent variable to solve for is the complex potential, ψ .

For purely harmonic waves, the following equation should hold in the whole domain:

$$\nabla\cdot\left(cc_g\nabla\psi\right)+k^2cc_g\psi = S \tag{3}$$

where c and c_g are the phase and group celerities, S is a source term presented in Subsection 2.3 and the modulus of k is given by the dispersion relationship.

$$\omega^2=kg\tanh(kh) \tag{4}$$

The dispersion relationship should be evaluated by expanding the hyperbolic term as shown in Goda (2000).

Once the depth (variable with x and y) and Omega (wave frequency) are defined, the following domain expression must be defined in order to find kh (the product of wavenumber and depth):

```
equ.expr = {'Kh2sw',
'depth*Omega^2/9.806',
'P','0.00011*Kh2sw^9+0.00039*Kh2s
w^8+0.00171*Kh2sw^7+0.00654*Kh2sw
^6+0.02174*Kh2sw^5+0.0632*Kh2sw^4
+0.16084*Kh2sw^3+0.3555*Kh2sw^2+0
.66667*Kh2sw+1',...
'kh','sqrt(Kh2sw^2+Kh2sw/P)'}
```

Iterative methods, even if virtually correct at any desired precision, are discontinuous, and henceforth less suited to a numerical minimisation procedure.

Equation 3 remains formally unchanged when the dependent variable is the surface elevation rather than the potential.

The velocity complex potential can be seen as:

$$\psi(x) = A(x)e^{i\chi(x)} \tag{5}$$

where $A(x)$ is the amplitude and $\chi=\boldsymbol{kx}$ is the phase, both real.

Oddly enough, inserting Eq. (5) into (3) we obtain the following eikonal equation:

$$\left(\nabla\chi\right)^2 = k^2 + o(cc_g, A) \tag{6}$$

which means that the gradient of the phase function $\boldsymbol{k}$ is not *exactly* the wavenumber given by the dispersion relationship and inserted in Eq. 3, but other negligible terms appear. Such small terms may have some effect only in case of strong convergence or divergence of the wave rays, and we shall not deal with them. On the contrary, we will derive the wavenumber vector (sometimes called the *effective* wavenumber) as the gradient of the phase function.

2.2 Boundary conditions

A single general boundary condition is considered, relative to partial reflection, which is function of a real valued reflection coefficient R.

Extreme cases can then be derived of total reflection, R=1, and total absorption (i.e. waves propagating out of the domain), R=0.

The boundary condition, assuming no phase shift between the incident and the reflected wave, is:

$$\Gamma\cdot\mathrm{n}=cc_g\frac{\partial\psi}{\partial n} = icc_gk\cos\beta\,\frac{1-R}{1+R}\psi \tag{7}$$

where β is the angle between the normal to the wave crest and the normal to the boundary n. It can be obtained following the iterative approach is suggested by Beltrami et al. (2001). The iterative procedure is not fully implemented here.

The initial guess for β assumed in the following is that the incident wave angle is equal to the nominal off-shore wave direction, with a lower limit (i.e. 10°).

In order to apply a correction on β, the wave direction must be evaluated. Theoretically, we could simply ask to Comsol Multiphysics to take the phase of the potential (i.e. obtain $\boldsymbol{kx}$ in the range 0-2π) and find its gradient. But this does not work, since the phase is not a simple function of the dependent variable.

A possible approach is to take the spatial derivative "manually" (i.e. outside the Comsol standard routines) by means of the `postinterp()` command.

A simpler solution comes from the following trick:

$$\nabla\psi = \nabla A e^{i\chi} + i\bar{k} A e^{i\chi}$$

$$\Rightarrow \bar{\mathrm{k}} = i\left(\frac{\nabla A}{A} - \frac{\nabla\psi}{\psi}\right) \qquad (8)$$

Here $\boldsymbol{k}$ is only function of spatial derivatives of φ and of its module. The wave directions, i.e. the components of $\boldsymbol{k}$, is shown in Figure 1, which is a benchmark case presented by Bellotti et al. (2003), a circular island on a paraboloidal shoal. Cosβ is merely the normalised product of $\boldsymbol{k}$ and the normal to the boundary.

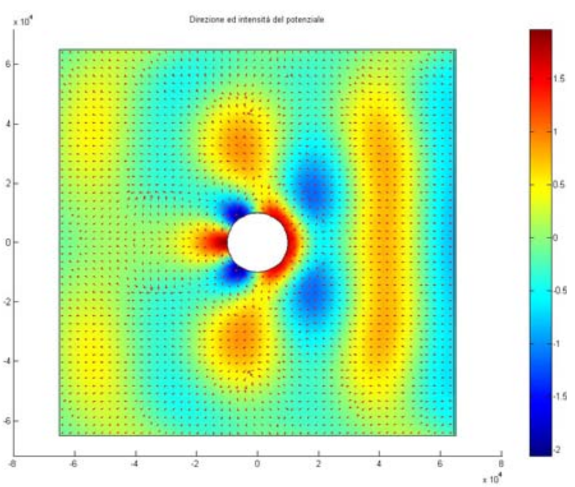

Figure 1. Waves propagating toward a sloping island. Cfr, Bellotti et al. (2003), fig. 6.

In Comsol Multiphysics, we shall have to set a few subdomain expressions. We first define χ as the phase of the dependent variable, and this is conveniently obtained by the following function:

$$\chi = \mathrm{unwrap}(\mathrm{atan2}(\psi, i\psi)) \qquad (9)$$

We should first ask to Comsol the gradient of χ and verify that we get a blank field rather than the correct answer (χ is seen to vary with regularity in space). We then obtain $\boldsymbol{k}=\{k_1,k_2\}$ as the gradient of χ by means of identity Eq. 8. For this purpose, we shall write the following 2 subdomain expressions:

$$\mathrm{k}_1 = i\,(|\psi|_x/|\psi| - \psi_x/\psi) \qquad (10)$$

$$\mathrm{k}_2 = i\,(|\psi|_y/|\psi| - \psi_y/\psi) \qquad (11)$$

where the subscript x and y stand for partial derivative of x or y.

The iterative procedure is more conveniently implemented by a script file.

2. Internal generation of waves

Bellotti et al. (2003) suggest a way to define the incident waves as if they were generated by an internal wavemaker.

It is otherwise difficult to define the incoming waves conditions. They would enter into the domain from an open boundary, where the exact potential is not known. Both reflected and scattered waves coexist with the incoming ones, but waves propagating back towards the open boundary are in fact not known, depending on the solution of the problem itself.

In order to generate waves from the interior of the domain, they derived a source term which is only active on a generation line and that inputs waves in both direction.

In order to switch on the term only on this line, a delta Dirac function δ(x) is used:

$$S = 2gc_g a\delta(x) \qquad (12)$$

where a is the amplitude of the generated wave.

This can be easily achieved in Comsol by considering S=0 in the domain eq. (3) and by adding a material line inside the domain, close to the off-shore boundary, where the internal boundary condition should not only insure continuity,

$$\mathrm{n}\,\Gamma_1 = \mathrm{n}\,\Gamma_2 \qquad (13)$$

but add a weak term equal to:

$$2\,\mathrm{i}\,c_g A_i\,\mathrm{test}(\psi) \qquad (14)$$

where A_i is the amplitude of the incoming wave, with frequency ω. This term will be computed as is added to the (left side of the) domain Eq. 3, but only on the line.

3 Model validation

The first validation must address the simple diffraction case, for regular waves.

Johnson (1952) developed diffraction coefficient diagrams for several regular wave cases, following an analytical approach. They can be found in the Shore Protection Manual (1984) and some of them also appear in the Coastal Engineering Manual (2002).

One example of such diagrams is shown in Fig. 2. It gives the diffraction of regular waves traveling with 15° obliquity toward a breakwater and passing through a gap of width equal to one wavelength.

This case has been modeled assuming all absorbing boundaries and results are given in Figure 3.

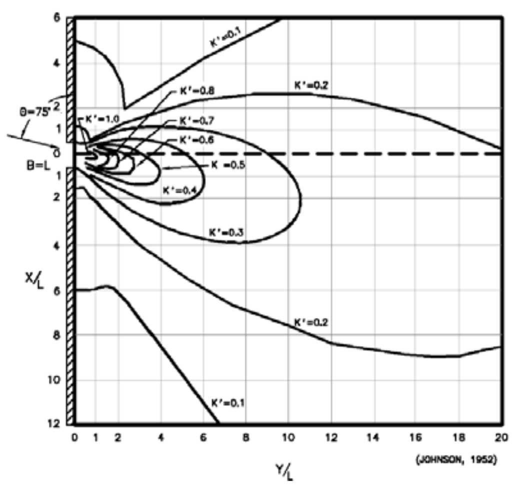

Figure 2. Diffraction coefficient for a wave incident with 15° obliquity on a breakwater and passing through a gap as wide as the wavelength.

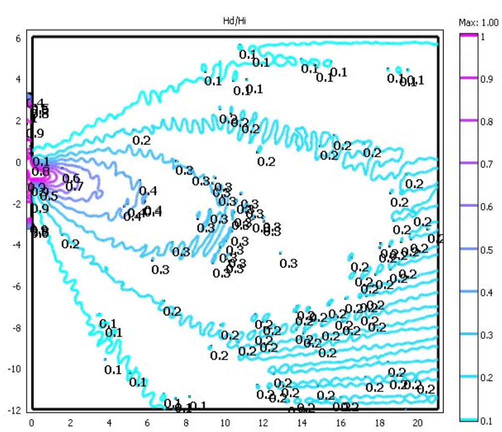

Figure 3. Simulated diffraction coefficient for the case in Fig. 2.

Contour lines are rather irregular since no smoothing was performed. The mesh, rather coarse, could not be refined due to severe limitations of the hardware. Nevertheless the quantitative agreement is already rather good and the convergence lines are easily identifiable.

One other model validations concerns the wave amplitude at the end of a long and reflective channel. Fig. 4 shows the comparison between the analytical solution proposed by Mei (1982) and the model outcomes, showing a very good agreement.

Details of the channel geometry are given in the paper by Bellotti et al. (2003). The same case is indeed used as benchmark in fig. 9 of their paper.

Resonance occurs when the channel length is 1/4 of the wavelength.

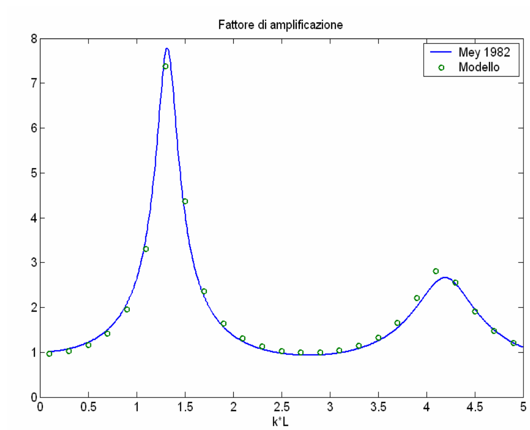

Figure 4. Wave amplification after propagation within a reflecting channel long *L*.

4. Example application

Casal Borsetti marina (Ravenna, Italy) frequently experiences high waves at the access channel. As a mere exercise, students of the 2008 course of Harbour Construction at University of Bologna, Faculty of Engineering, held by Prof. A. Lamberti, were asked to design some pier extension to allow a smoother entrance to the marina.

Table 1. Incident wave climate (off-shore)

Hs [m]/ dir [°]	calma	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	TOT%
0		1,06	1,37	0,35	0,19	0,03	0,03	-	0,01	-	-	**3,04**
30		1,26	1,96	0,94	0,93	0,21	0,09	0,06	0,03	0,01	0,02	**5,51**
60		2,29	3,05	1,39	1,23	0,29	0,18	0,08	0,03	0,01	0,03	**8,58**
90		5,21	5,76	2,15	1,41	0,34	0,36	0,06	0,01	0,01	0,01	**15,31**
120		2,48	3,38	0,95	0,63	0,11	0,10	0,02	-	-	-	**7,68**
150		1,42	2,31	0,47	0,29	0,02	0,02	-	-	-	-	**4,54**
180		0,94	1,48	0,23	0,06	-	-	-	0,01	-	-	**2,73**
210		1,31	1,41	0,17	0,04	0,01	-	-	-	-	-	**2,93**
240		3,19	2,62	0,23	0,06	0,01	0,02	-	0,01	-	-	**6,15**
270		8,49	6,33	0,49	0,16	0,02	0,06	0,01	0,02	0,01	-	**15,60**
300		4,90	4,56	0,49	0,10	0,01	0,02	-	0,01	-	-	**10,08**
330		3,04	3,32	0,46	0,11	0,01	0,01	-	-	-	-	**6,94**
TOT%	**10,90**	**35,60**	**37,55**	**8,34**	**5,21**	**1,05**	**0,88**	**0,24**	**0,13**	**0,05**	**0,06**	

Figure 5. Access channel to Casal Borsetti Marina. The harbour extension in given with a gray line.

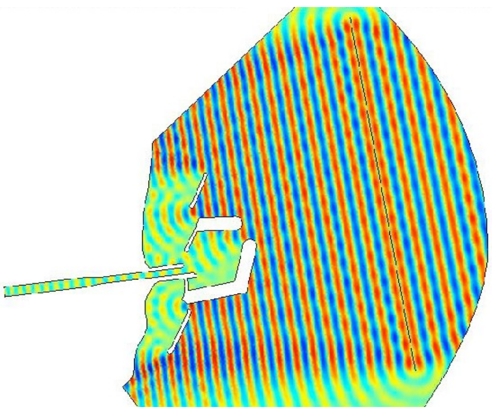

Figure 6. Example of wave propagation (surface elevation in colour scale). The generation line is visible on the right, almost perpendicular to the access channel.

Table 2. Wave height at the port entrance channel, for each condition shown in Table 1

	Ho	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5
An-golo	θrel										
30	-70	Ver.	0,343	0,567							
60	-40	Ver.	Ver.	Ver.	0,387						
90	-10	Ver.	Ver.	Ver.	Ver.	0,344					
120	20	Ver.	Ver.	Ver.	Ver.	0,254	0,39				
150	50	Ver.	Ver.	Ver.	Ver.	Ver.	Ver.	Ver.	Ver.	Ver.	0,199

The design is given in Figure 5 in gray color, superimposed to an aerial picture. The bottom topography was obtained as interpolation of a table with depths.

The wave conditions are given in Table 1. For each wave direction and wave period, a simulation adopting this model was run in order to see the internal wave agitation. The generation line was rotated in order to be perpendicular to the wave direction. An example of output is given in Fig. 6.

For each wave condition, the highest wave in front of the straight channel is selected. Table 2 shows when the channel can be navigated (white cells). Most of the times with high waves, breakers impede navigation anyway (blue cells). For the class with waves from bora 30°, height 1.5 m, navigation is possible and yet the waves slightly exceed the target value so that access to the port is not guaranteed. As a fact, most of the marina in that area are open toward south.

5. Conclusions

The proposed model, validated against several benchmarks and applied to a realistic yet fictitious case, is seen to be suited for the study of the harbour response to external waves likewise other specific dedicated software.

6. References

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7. Acknowledgements

We wish to thank Dr. G. Bellotti for his precious suggestions, and acknowledge the students who designed the structure in Fig. 5: Daniele Polverelli, Fabio Venturini, Daniele Romandini, Natxco Villalba.