Convergence Rates for Models with Combined 1D/2D Subdomains

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Abstract: It is well known that the convergence rate of a numerical model is significantly reduced, when the genuine character of the set-up with differential equation and boundary condition is not given anymore. Here we examine the decrease of convergence order for models in which the problem set-up includes a combination between a 1D and a 2D subdomain. In terms of physics we are dealing with flow in the vicinity of a fracture in a porous medium.

We study two different set-ups, one within a (potentially) unbounded domain, and one in a bounded domain between two impermeable strata. We examine the convergence rates for different physical parameter sets, for different finite elements, and different error norms. It turns out that the convergence rate is significantly reduced. Moreover it turns out to be independent of element order – an indication that the combined geometry and not the finite element discretisation is the bottleneck in this type of model.

Keywords: convergence rate, finite element order, potential equation

1. Introduction

This is an extension of the work on convergence rates presented by Bradji & Holzbecher at former COMSOL conferences (Bradji & Holzbecher 2007, 2008). In the first paper we studied mainly simple set-ups, as the potential and Poisson equations, using the maximum norm. In the second paper we extended our work using other norms in addition, like the average and the energy norms. In both papers we give a detailed description, how we derive convergence rates for a series of irregular free meshes, which we do not recall here.

In our former work we found that the convergence rate is significantly reduced, if the problem set-up deviates from the 'ideal' constellation, for which theoretical results concerning convergence rate are available. For example, in a potential problem with both Dirichlet- and Neumann boundary conditions we obtained 1st order convergence only (in maximum norm), instead of the higher order observed in problems with only Dirichlet boundaries. Here it is the intention to extend the study to problems where 1D and 2D geometries are combined. As expected, we also observe reduced convergence rates for such a situation.

For the mentioned purpose we examine two constellations of a fracture in a porous medium. In one set-up the fracture is located in a free in principle unbounded flow field within the porous medium. In the other set-up the flow field is sandwiched between two impermeable strata.

2. Model Set-ups

We examine two set-ups, one with free flow and one with a bounded flow field. Set-up 1 has a horizontally placed fracture in a diagonal flow field. The geometry and extension of the model region is depicted in Figure 1. The same constellation was studied by Sato (2003) and by Churchill & Brown (1984).

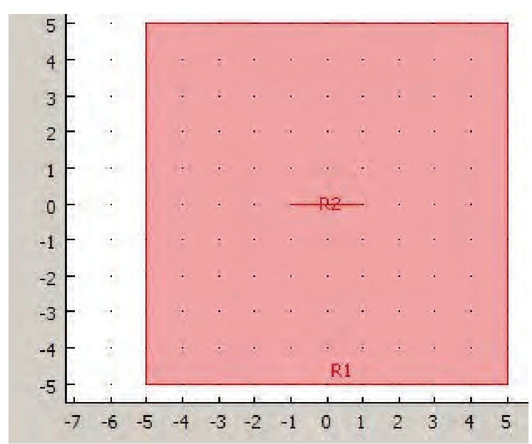

Figure 1. Fracture (R2) in a free flow field (set-up 1)

In the 2D model region R1 we require the differential equation

$$\nabla K_{low} \nabla \varphi = 0 \tag{1}$$

for the hydraulic potential φ to be fulfilled. In the 1D (or 2D) region R2 we require:

$$\nabla K_{high} \nabla \varphi = 0 \tag{2}$$

Conditions at the outer boundaries of R1 are of Dirichlet type throughout:

$$\begin{aligned} \varphi(x,5) &= 0.5 - .05*(x+5) \\ \varphi(x,-5) &= 1 - .05*(x+5) \\ \varphi(5,y) &= 0.5 - .05*(y+5) \\ \varphi(-5,y) &= 1 - .05*(y+5) \end{aligned} \tag{3}$$

With these we specify diagonal inflow with an angle of 45° from the lower left part. If a 1D approach for the fracture is used, the boundary condition on the two end points of interval R2 is:

$$\frac{\partial \varphi}{\partial x}(\pm 1, 0) = 0 \tag{4}$$

If we model R2 as 1D, this is a combined 1D/2D problem! A formulation for combinations of 1D, 2D and 3D subdomains in general is given by Sembera *et al*. (2007), who also propose a finite element model approach for such situations. Similarly Angot *et al*. (2009) compare 'global Darcy models' with 'asymptotic' approaches. Very recently Grillo *et al*. (to appear) deal with lower-dimensional approaches for density-driven flow in fractured porous media.

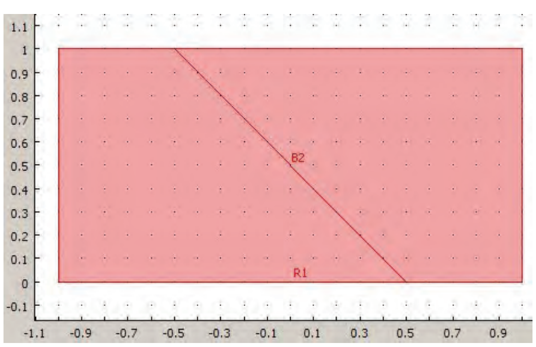

Figure 2. Fracture (B2) in a bounded flow field (set-up 2)

Set-up 2 has a diagonally placed fracture in a flow field that is bounded two sides. The geometry and extension of the model region is depicted in Figure 2. Lower and upper horizontal lines represent no-flow boundaries. The flow is induced by a pressure, resp. head gradient between the vertical boundaries, inducing flow from left to right. Thus there are boundary conditions of Dirichlet- and of Neumann type.

For the parameter variations we used $K_{low} = 1/K_{high}$ and obtain for the ratio between the conductivities:

$$K_{ratio} = \frac{K_{high}}{K_{low}} = K_{high}^2 \tag{5}$$

Note that we use normalized values for length and velocity in both models. A real world problem with reference length L and velocity v_0 we can transfer to the normalized system by the transformations $x \to x/L$ and $y \to y/L$ and normalized velocity $v \to v/v_0$. For the given choice of conductivities we obtain a formula for the reference velocity as a function of fracture conductivity K_f and matrix conductivity K_m of the real system

$$v_0 = \sqrt{K_m K_f} \quad \text{or} \quad v_0 = K_{ratio}\sqrt{K_m} \tag{6}$$

Concerning analytical solutions for set-up 1 and more general remarks on fracture modeling see Holzbecher *et al.* (2010).

3. Use of COMSOL Multiphysics

The models were set-up using COMSOL Multiphysics 3.5a. We use the diffusion mode for the solution of equations (1) and (2), with diffusivities K_{low} and K_{high} respectively. The potential values, calculated for the fracture in the 1D fracture geometry, are used as boundary conditions in the 2D geometry. For that purpose we use the COMSOL option to enter 'extrusion coupling variables'. Conditions for the outer boundaries in set-up 1 are entered according to formulae (3) and (4).

For intercomparison we also modeled the two set-ups by models with complete 2D geometry with matrix and fracture as two subdomains. There is no coupling necessary: the conductivity parameter changes from one subdomain to the other. The fracture is resolved by a very fine 2D mesh.

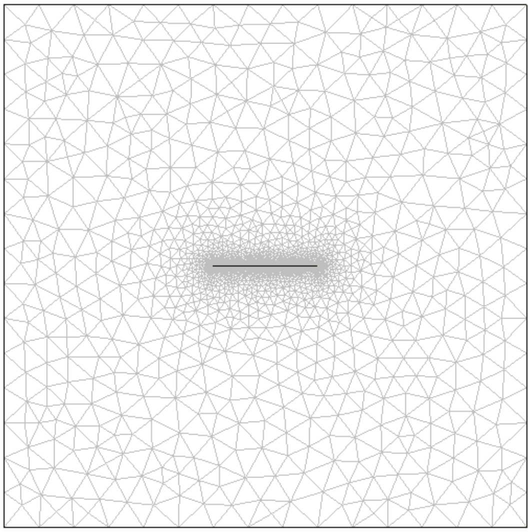

Figure 3. Finite element mesh of set-up 2

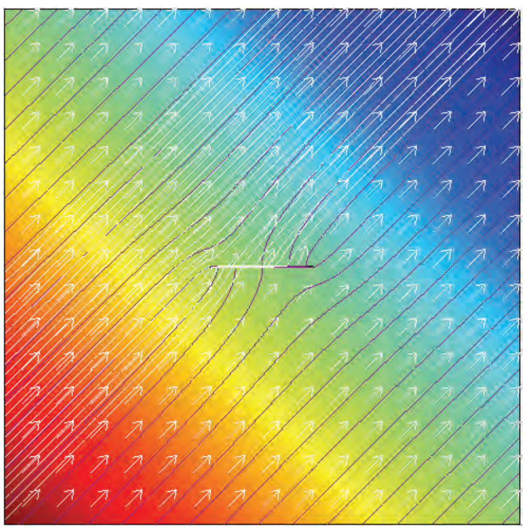

Figure 4. Potential colorplot, streamlines (magenta), flowpaths (white) and velocity field (white) for set-up 1

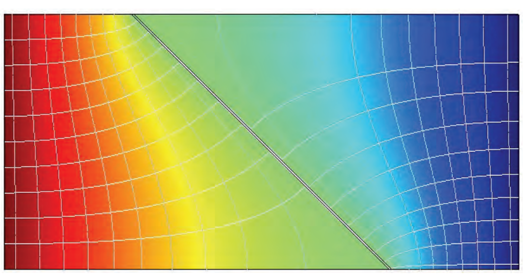

Figure 5. Potential colorplot and streamlines for set-up 2 (K_{low}=0.01, K_{ratio}=10000)

Meshing was always made with a drastic refinement towards the fracture. As an example we show a mesh for the free fluid situation with 3448 elements and 1755 mesh points in Figure 3. Using quadratic Lagrange elements this corresponds to 14396 degrees of freedom (DOF). For the convergence studies meshes with much higher DOFs have been used (see appendix).As reference for the calculation of the error we use model runs with extremely fine meshes. For the 2D fracture of set-up 1 for example, we started with the 'extremely fine' option and maximum element size of 0.001 in the fracture, before after two refinements, ending up with a mesh resulting in 3240961 DOFs.

4. Results

A graphical representation of the results for set-up 1 is given in Figure 4.

In the studies of convergence rates we used the K_{high}=100 corresponding to K_{ratio}=10000. The result for this parameter in set-up 2 is given in Figure 5.

Altogether we worked with 6 meshes for 2nd order elements and 7 meshes for 1st order elements. The coarsest mesh corresponds with 715 DOF for set-up 1 (210 for set-up 2) in case of 1st order elements, and 2577 DOF (756) for 2nd order elements. The finest mesh corresponds with ≈2.5 Mio. DOF for set-up 1. Convergence rates ϑ are calculated in the same manner as outlined by Bradji & Holzbecher (2007, 2008).

Details on all DOFs in intermediate meshes and errors (denoted by $\|e\|$), as well as resulting convergence rates for all refinements are given in the appendix. Here we gather the mean convergence rates in Tables 1 for set-up 1 and Table 2 for set-up 2.

Errors were calculated in relation to a reference solution obtained for very fine finite element meshes. Some details are given in the previous section.

Following Bradji & Holzbecher (2007) and Bradji & Holzbecher (2008) one has to compare the obtained convergence rates with the theoretical values of 3.0 for 2nd order elements in the maximum and average norm, with 2 for 2nd order elements and energy norm; with 2.0 for maximum and average norm for 1st order elements and with 1.0 for 1st order elements and the energy norm.

Table 1: Convergence rates for combined 1D/2D problem (set-up 1)

	1st order elements	**2nd order elements**
Maximum norm	0.5	0.5
Average (L2) norm	1.0	1.0
Energy norm	0.5	0.5

Table 2: Convergence rates for combined 1D/2D problem (set-up 2)

	1st order elements	**2nd order elements**
Maximum norm	0.7	0.7
Average (L2) norm	1.0	1.0
Energy norm	0.5	0.5

Thus the convergence is much worse than the theory predicts for generic models. The convergence rates are even lower than for problems with mixed Dirichlet- and Neumann-boundary conditions (Bradji & Holzbecher 2007).

For comparison we also determined the convergence rates for the corresponding problems, in which the fracture is modeled in 2D, using an increased hydraulic conductivity. The convergence rates for that modeling approach, depending on element order and norm, are given in Table 3.

Table 3: Convergence rates for 2D problem (set-up 1)

	1st order elements	**2nd order elements**
Maximum norm	0.72	0.78
Average (L2) norm	1.78	1.73
Energy norm	1.20	0.75

Again 6 meshes with increased refinement have been taken into account. The values in Table 3 are the mean values resulting from those runs. Depending on the norm the convergence rates may vary quite drastically. For the maximum norm the variation between 0.65 and 0.99 is very moderate. It is higher for the average norm, between 1.33 and 1.95 for 1st order elements and between 0.84 and 2.59 for 2nd order elements. Highest variation is for the energy norm between 0.5 and 1.7 for 1st order elements and between 0.47 and 1.21 for 2nd order elements.

For all norms and elements the convergence rates of the 2D model are higher than in the combined 1D/2D model, but they are also significantly reduced in comparison to the theoretical expectation. It is also an interesting result that here also the element order has no influence on the convergence rate.

The high variability of convergence rates can surely be attributed to the ‘heterogeneity’ of the set-up, if entirely discretized in 2D. The global mesh characteristics are minor important. What makes a difference is the mesh refinement within the fracture, for which the number of DOFs in the entire model region is surely not a consistent measure.

5. Conclusions and Perspective

To our knowledge there are no results concerning the theoretical convergence order for the finite element approximation for the models treated in the present work. Therefore, the present contribution, which provides the numerical convergence order for the COMSOL solution of some Fracture Models, can be considered as useful information about the expected theoretical convergence order of the finite element approximation for the Fracture models. We think that the present contribution is a starting step to collect the existing literature concerning both theoretical and numerical convergence of the finite element approximation for the Fracture models.

The convergence rates for the combined 1D/2D model are significantly reduced in comparison to the pure 2D-set-up. Moreover there seems to be no dependence of the finite element order. This is a clear indication that the finite element discretisation is not the crucial part of the numerical solution. The coupling between the 1D and 2D domains most likely is the limiting process in the entire model constellation that has a substantial influence on the convergence.

The pure 2D discretisation also shows a reduced convergence rate in comparison to the more 'homogeneous' testcases, tackled by Bradji & Holzbecher (2007) and Bradji & Holzbecher (2008). However all other characteristics of the convergence behavior remain valid. In the energy norm the performance is worse than in the average norm, while using maximum or average norm provides similar results (for 2. order elements). The model thus simply performs worse on all criteria, which has to be attributed to the 'inhomogeneous' set-up of the model, i.e. the change of model properties in some parts by several orders of magnitudes.

6. References

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7. Appendix

Tables 7-12: Details of convergence order evaluation for combined 1D/2D model for fracture in set-up 1

2^{nd} order elem.; max-norm

DOF	$\|e\|\cdot 10^2$	ϑ
2577	9.2220	0.5039
9999	6.5535	0.5090
39387	4.6231	0.5398
156339	3.1867	0.5684
622947	2.1513	0.5345
2486979	1.4861	

2^{nd} order elem.; L2-norm

DOF	$\|e\|\cdot 10^2$	ϑ
2577	5.5346	1.0643
9999	2.6900	1.0421
39387	1.3168	1.0418
156339	0.6422	1.0639
622947	0.3078	1.1243
2486979	0.1414	

2^{nd} order elem.; energy norm

DOF	$\|e\|\cdot 10^2$	ϑ
2577	3.0367	0.5269
9999	2.1246	0.5092
39387	1.4986	0.4984
156339	1.0629	0.4907
622947	0.7571	0.4879
2486979	0.5401	

1^{st} order elem.; max-norm

DOF	$\|e\|\cdot 10^2$	ϑ
715	1.4737	0.4647
2638	1.0881	0.4846
10120	0.7856	0.5052
39628	0.5565	0.5076
156820	0.3925	0.5744
623908	0.2640	0.5851
2488900	0.1761	

1^{st} order elem.; L2-norm

DOF	$\|e\|\cdot 10^2$	ϑ
715	1.7335	1.0772
2638	0.8582	1.0418
10120	0.4260	1.0273
39628	0.2113	1.0265
156820	0.1043	1.0393
623908	0.0509	1.0741
2488900	0.0242	

1^{st} order elem.; energy norm

DOF	$\|e\|\cdot 10^2$	ϑ
715	5.6702	0.5776
2638	3.8891	0.5383
10120	2.7082	0.5169
39628	1.9031	0.5045
156820	1.3451	0.4962
623908	0.9549	0.4786
2488900	0.6857	

Tables 13-18: Details of convergence order evaluation for combined 1D/2D model for fracture in set-up 2

2^{nd} order elem.; max-norm

DOF	$\|e\|\cdot 10^2$	ϑ
756	2.2204	0.7102
2862	1.3839	0.6983
11130	0.8614	0.6971
43890	0.5340	0.7144
174306	0.3263	0.7560
694722	0.1935	

2^{nd} order elem.; L2-norm

DOF	$\|e\|\cdot 10^2$	ϑ
756	6.5556	1.0329
2862	3.2962	1.0242
11130	1.6443	1.0339
43890	0.8090	1.0682
174306	0.3873	1.1512
694722	0.1747	

2^{nd} order elem.; energy norm

DOF	$\|e\|\cdot 10^2$	ϑ
756	8.4163	0.4950
2862	6.0537	0.4916
11130	4.3354	0.4933
43890	3.0907	0.5015
174306	2.1871	0.5216
694722	1.5249	

1^{st} order eleme.; max-norm

DOF	$\|e\|\cdot 10^2$	ϑ
210	5.5762	0.7770
756	3.3901	0.7243
2862	2.0934	0.7010
11130	1.3005	0.7012
43890	0.8039	0.7010
174306	0.4958	

1^{st} order elem.; L2-norm

DOF	$\|e\|\cdot 10^2$	ϑ
210	2.3953	1.0073
756	1.2566	0.9967
2862	0.6472	0.9960
11130	0.3291	1.0039
43890	0.1653	1.0226
174306	0.0816	

1^{st} order elem.; energy norm

DOF	$\|e\|\cdot 10^2$	ϑ
210	1.6479	0.5162
756	1.1840	0.4962
2862	0.8510	0.4888
11130	0.6106	0.4882
43890	0.4369	0.4952
174306	0.3105	