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W I A S

Weierstraß-Institut für Angewandte Analysis und Stochastik

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Accuracy Tests for COMSOL - and Delaunay Meshes

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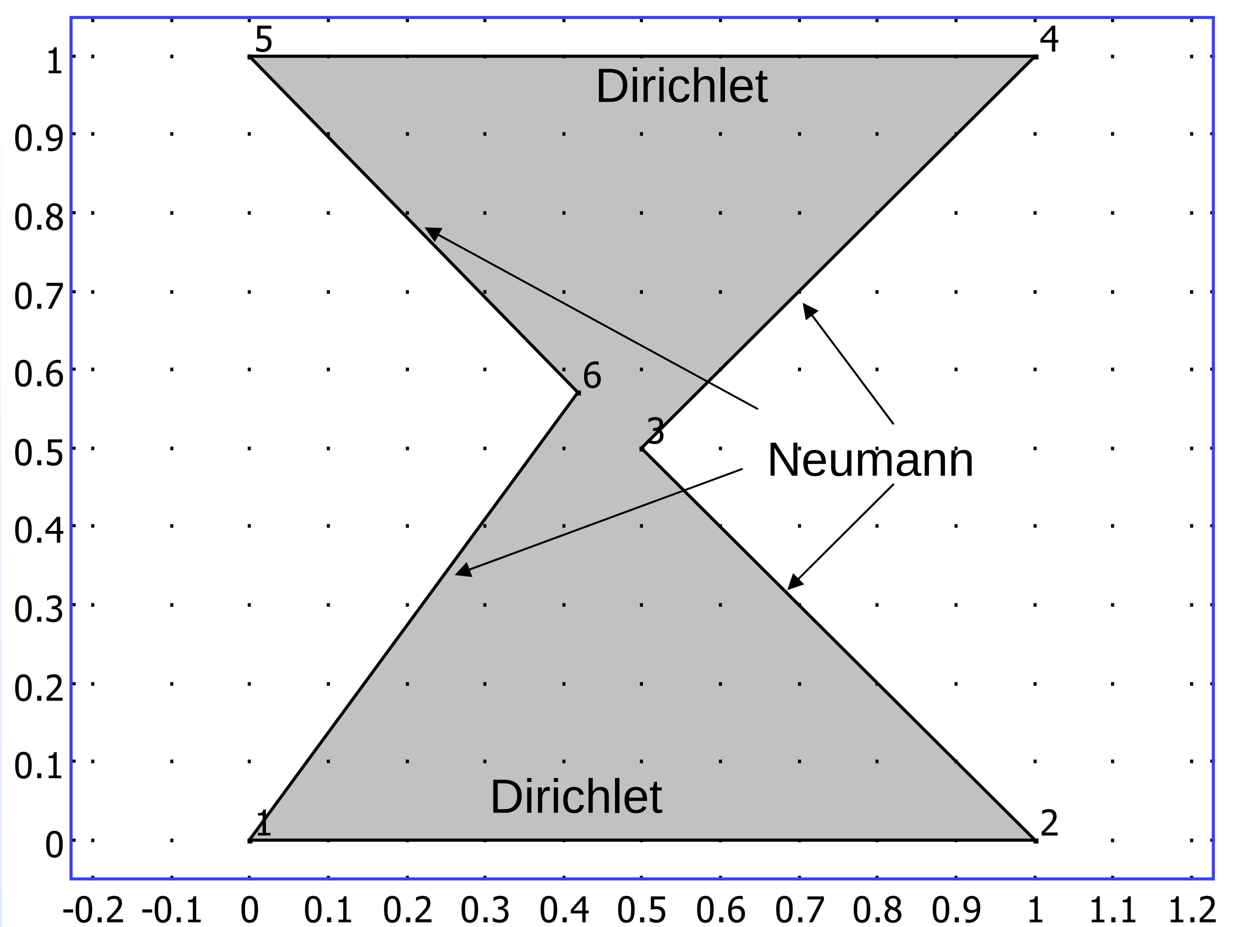
Mesh Test - General

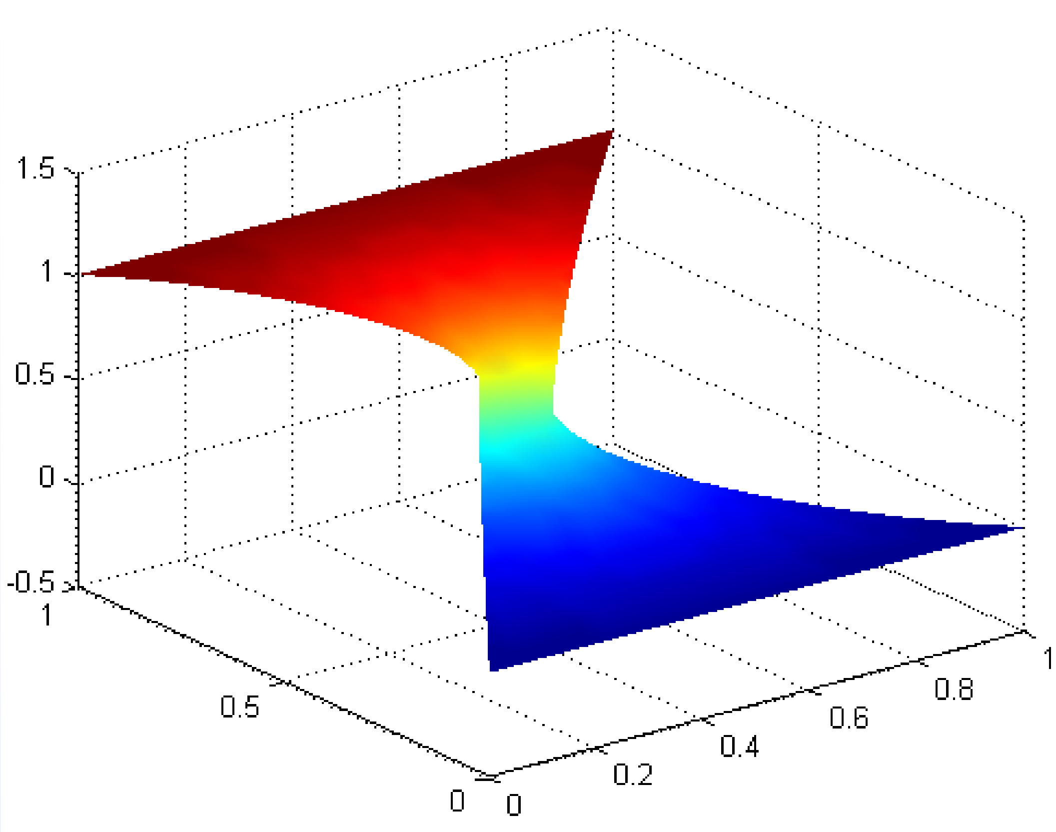
- We test the influence of the mesh on the accuracy of a COMSOL Finite Element solution
- We choose 2D and 3D testcases
  - with classical differential equation
  - and a complex geometry
- We compare linear and quadratic elements
- We study regular mesh refinement and adaptive mesh refinement
- We study meshes with and without Delaunay property

TestCase 1, Definition



2D single subdomain, potential equation: $\nabla^2 u = 0$

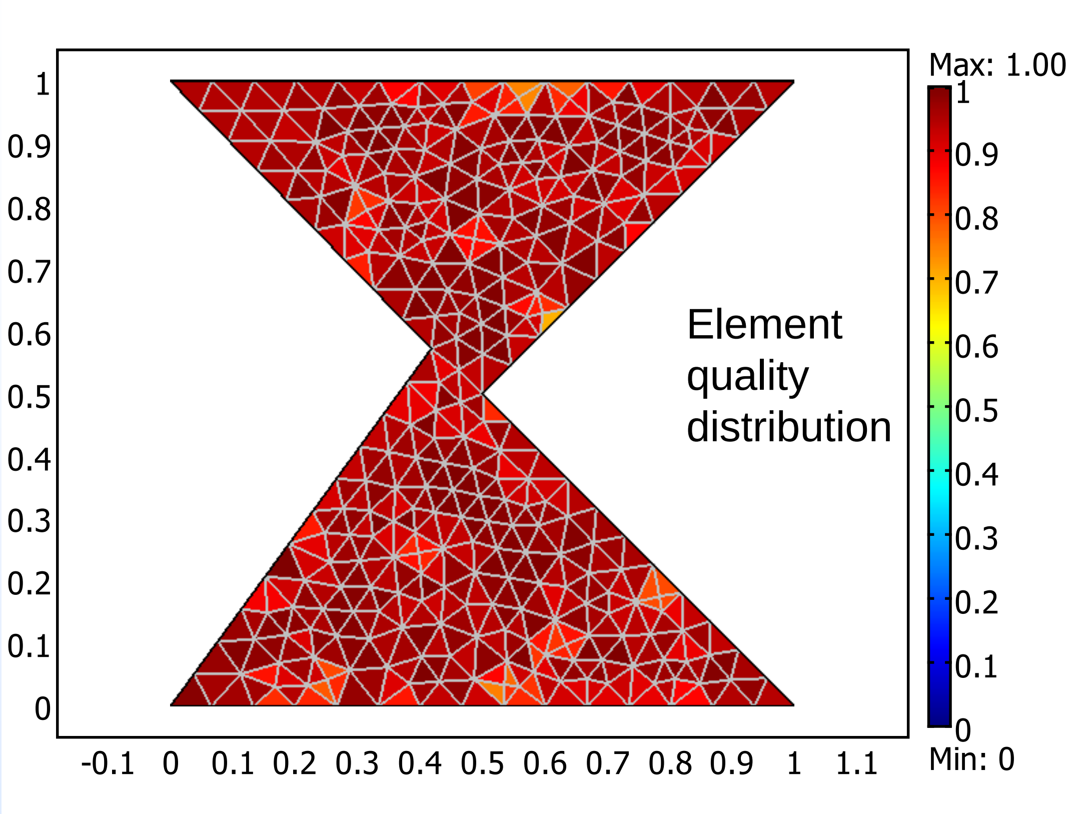
TestCase 1, Analytical Solution



- Analytical solution by Schwarz-Christoffel Transformation
- using MATLAB SC*-toolbox by Driscoll & Trefethen

* Schwarz-Christoffel

TestCase 1, Mesh Quality (2D)



Element quality: $q = \frac{4\sqrt{3}A}{h_1^2 + h_2^2 + h_3^2}$

with: area A and sidelengths h_1, h_2 and h_3

Mesh quality is defined as the minimum element quality

TestCase 1, Results for Quadratic Elements

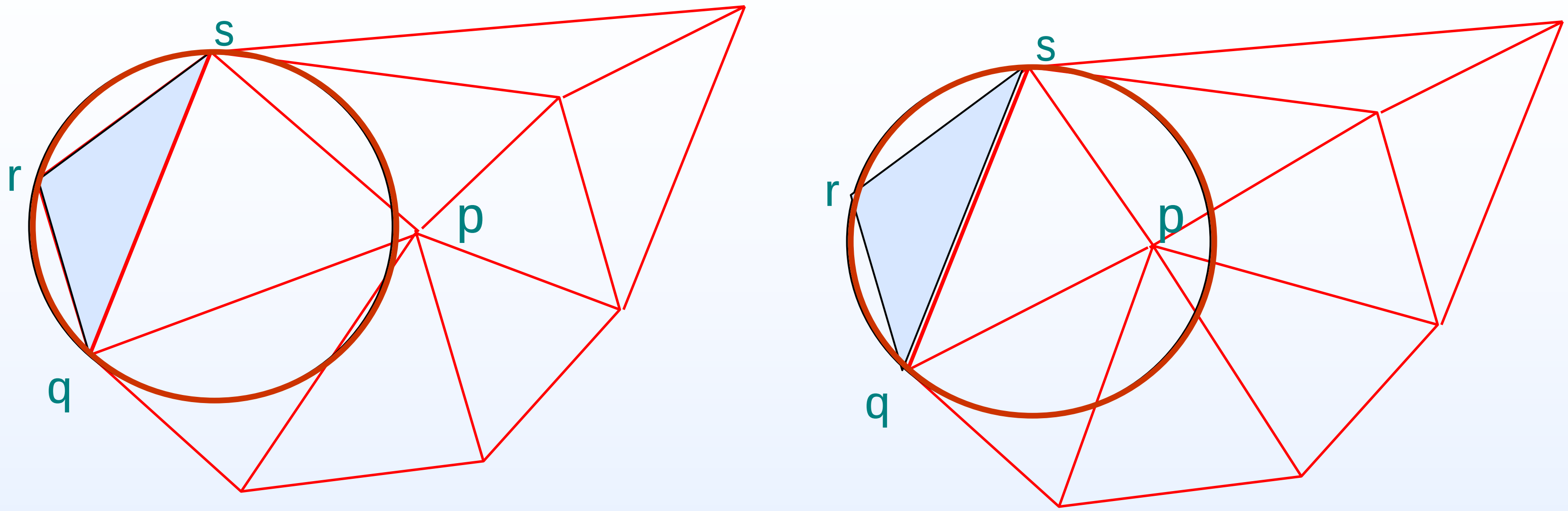
Refine-ments	DOF	No. elements	$10^4 \|e\|_2$	ϑ	
0	1085	506	301	1.25	
1	4193	2024	129		0.91
2	16481	8096	69	0.94	
3	65345	32384	36		2.97
4	260225	129536	13		

Degrees of freedom — Convergence order

with convergence order defined by $$\vartheta = -2\frac{\ln\left(\|e_1\|\right) - \ln\left(\|e_2\|\right)}{\ln\left(DOF_1\right) - \ln\left(DOF_2\right)}$$

Quadratic elements; refinements are regular.

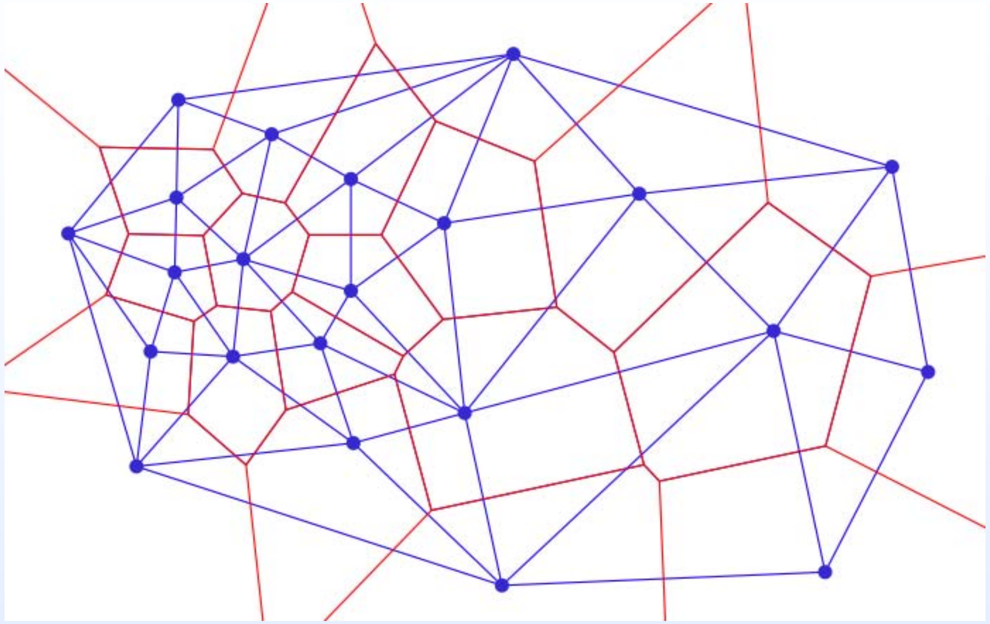
Delaunay Meshes



The **Delaunay triangulation** is defined by the property that there are no further nodes within the circumspheres of the triangles

(Delaunay 1934, russ.)

Voronoi Diagrams



-- Delaunay triangulation

-- Voronoi diagram

In the **Voronoi diagram** each cell consists of points closest to one node

The Voronoi diagram is the dual representation of the Delaunay triangulation;

TestCase 1, Delaunay Meshes (Quadratic Elements)

Delaunay meshes, produced with ‚triangle' (Shewchuk)

Mean elem. size	**DOF**	**# elem.**	$\|e\|_2$ **10⁴**
10^{-3}	1764	833	26897
$10^{-3}/2$	3526	1693	126
$10^{-3}/4$	6950	3375	101
$10^{-3}/8$	13840	6783	78

Default option

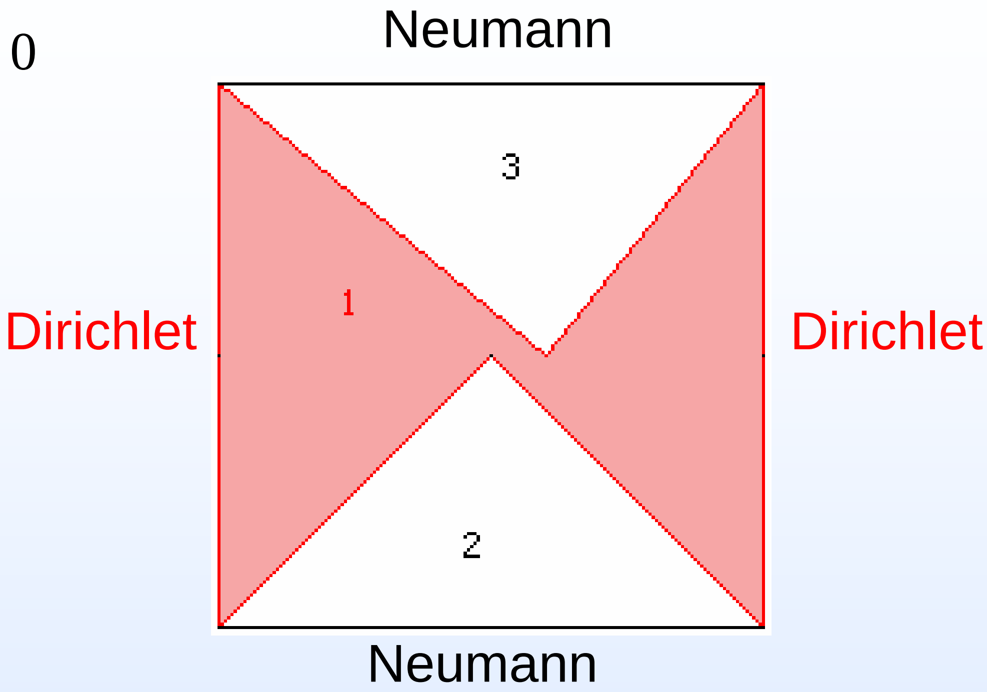
Mean elem. size	**DOF**	**# elem.**	$\|e\|_2$ **10⁴**
10^{-3}	1849	854	177
$10^{-3}/2$	3535	1678	133
$10^{-3}/4$	6977	3352	102
$10^{-3}/8$	13805	6728	73
$10^{-3}/8^{*}$	14257	6954	75

D option (improved quality)

* q option (30° angle restr.)

TestCase 2, Definition

$$\nabla\left(\sigma\nabla u\right)=0$$



- 2D three subdomain set-up
- High permeability (diffusivity) in domain 1 (1)
- Low permeability (diffusivity) in domains 2 and 3 (10^{-4} and 10^{-5})

TestCase 2, Results 1

Regular refinement

Linear elements

<table>
<tr><th>Refine ments</th><th>DOF</th><th># elem.</th><th>$\|e\|$ 10^4</th><th colspan="2">ϑ</th></tr>
<tr><td>0</td><td>500</td><td>938</td><td>2302</td><td rowspan="2">1.22</td><td></td></tr>
<tr><td>1</td><td>1937</td><td>3752</td><td>996</td><td rowspan="2">1.23</td></tr>
<tr><td>2</td><td>7625</td><td>15008</td><td>433</td><td rowspan="2">1.21</td></tr>
<tr><td>3</td><td>30257</td><td>60032</td><td>188</td><td rowspan="2">1.25</td></tr>
<tr><td>4</td><td>120545</td><td>240128</td><td>79</td><td></td></tr>
</table>

Quadratic elements

<table>
<tr><th>Refine ments</th><th>DOF</th><th># elem.</th><th>$\|e\|$ 10^4</th><th colspan="2">ϑ</th></tr>
<tr><td>0</td><td>1937</td><td>938</td><td>593</td><td rowspan="2">1.22</td><td></td></tr>
<tr><td>1</td><td>7625</td><td>3752</td><td>256</td><td rowspan="2">1.23</td></tr>
<tr><td>2</td><td>30257</td><td>15008</td><td>110</td><td rowspan="2">1.33</td></tr>
<tr><td>3</td><td>120545</td><td>60032</td><td>44</td><td rowspan="2">1.65</td></tr>
<tr><td>4</td><td>481217</td><td>240128</td><td>14</td><td></td></tr>
</table>

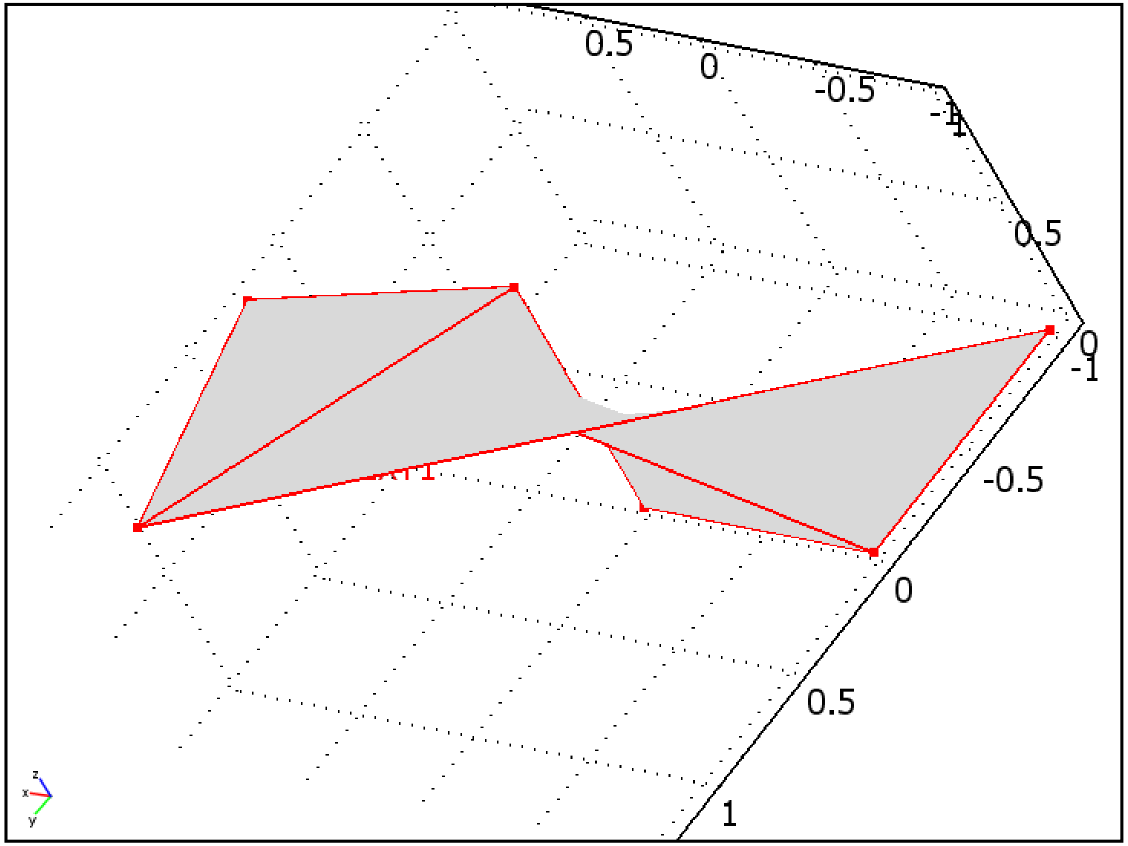
TestCase 2, Results 2

Adaptive refinement

Refine-ments	DOF	No. elements	Mesh increase	$\|e\|$
1	1997	968	1.032	284
2	2089	1014	1.048	133
3	2133	1036	1.022	125
4	2185	1062	1.025	64
5	2301	1120	1.055	55
6	2401	1170	1.045	35
7	2517	1228	1.050	24
8	2577	1258	1.024	21
9	2773	1356	1.078	20
10	2973	1456	1.074	12
11	3193	1566	1.076	10
12	3505	1722	1.100	10
13	3829	1884	1.094	10

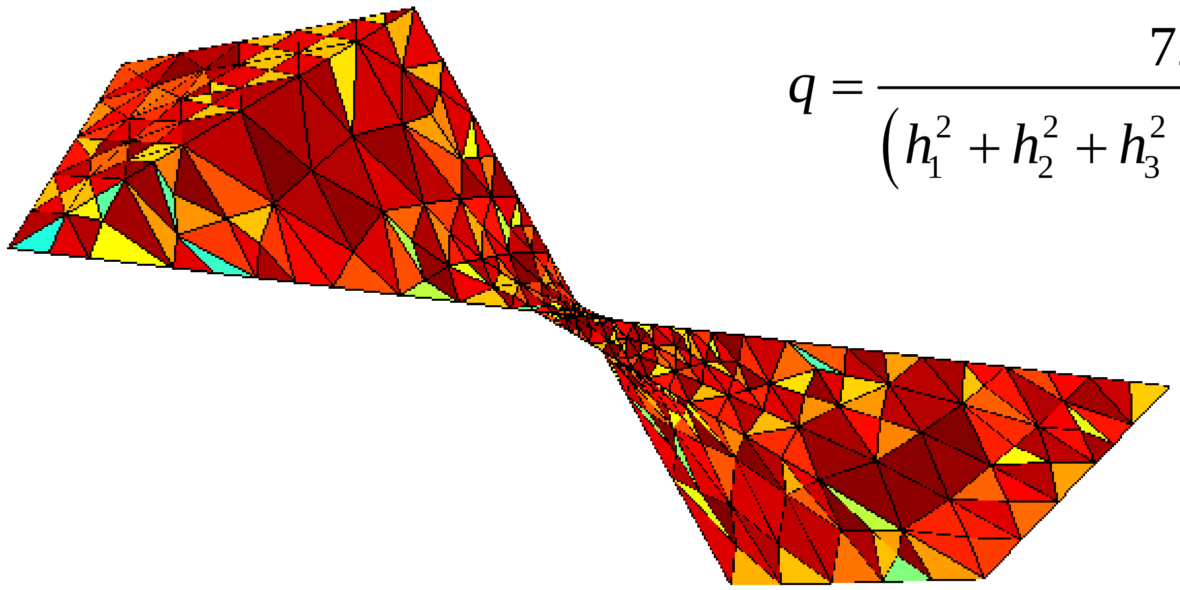
Quadratic elements,
residual method: coefficient,
refinement method: regular,
element selection method: fraction of worst error (parameter 0.5)

TestCase 3 (3D), Set-up



The 3D domain is produced by performing a shift and a rotation on a triangle simultaneously. The angle is 165°.

TestCase 3 (3D), Mesh



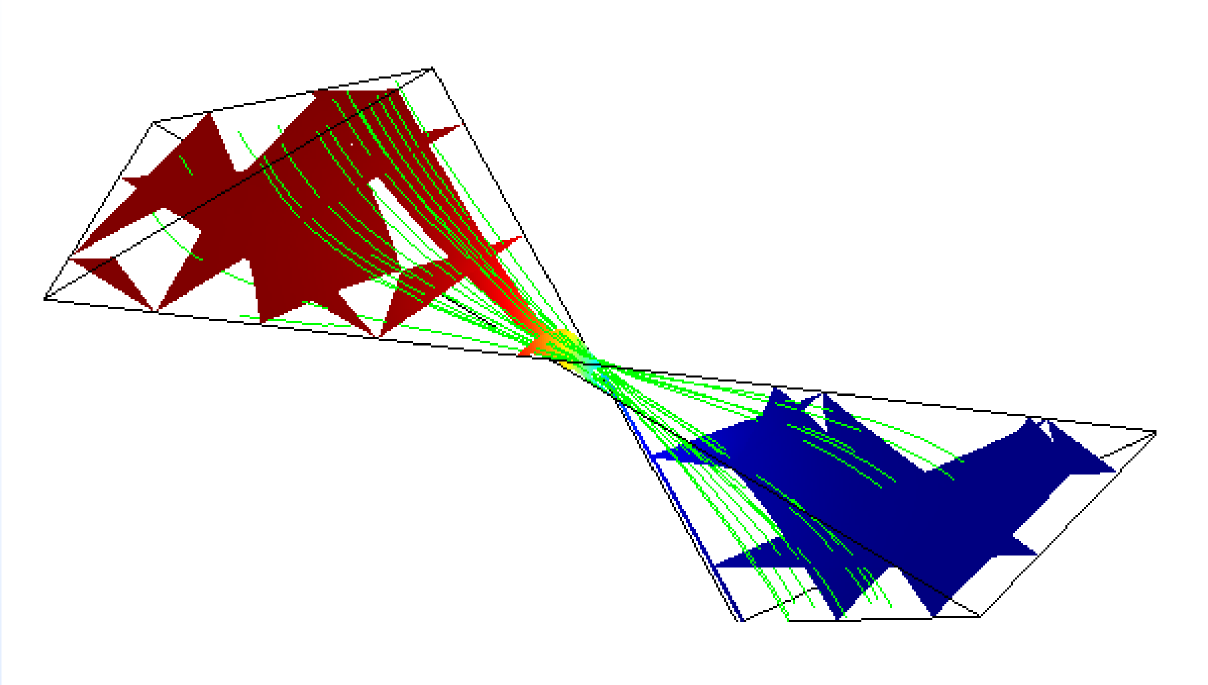
$$q = \frac{72\sqrt{3}V}{\left(h_1^2 + h_2^2 + h_3^2 + h_4^2 + h_5^2 + h_6^2\right)^{3/2}}$$

2465 elements

Element quality from 0 (blue) to 1 (red)

Mesh quality: 0.1174

TestCase 3 (3D), Solution



- Laplace equation
- Dirichlet conditions at the two ‚end'-positions of the triangle
- Neumann conditions at all other boundaries

TestCase 3 (3D), Results for Linear Elements

Mesh	**DOF**	**Quality optim.**	**No. elements**	**Mesh quality**	$\|e\|_2$ **10²**
extra coarse	74	-	162	0.0846	2.21
coarser	108	-	253	0.0508	2.21
coarse	140	-	352	0.0567	1.46
normal	255	-	714	0.0561	0.55
fine	464	-	1492	0.0248	0.42
finer	1028	-	3849	0.0270	0.25
extra fine	2726	-	11801	0.0192	0.127
extra coarse	75	+	161	0.2030	2.58
coarser	109	+	250	0.1934	0.90
coarse	141	+	342	0.0697	1.47
normal	256	+	676	0.1814	0.60
fine	464	+	1407	0.2049	0.38
finer	1028	+	3633	0.1954	0.26
extra fine	2726	+	11063	0.2191	0.109

TestCase 3 (3D), Results for Quadratic Elements

Refine-ments	DOF	Quality optim.	No. elements	Mesh quality	$\|e\|_2$ 10^2
extra coarse	381	-	162	0.0846	16.8
coarser	573	-	253	0.0508	12.5
coarse	765	-	352	0.0567	9.72
normal	1461	-	714	0.0561	2.28
fine	2821	-	1492	0.0248	0.79
finer	6668	-	3849	0.0270	0.22
extra fine	18896	-	11801	0.0192	0.09
extra coarse	382	+	161	0.2030	17.9
coarser	572	+	250	0.1934	12.2
coarse	757	+	342	0.0697	9.61
normal	1425	+	676	0.1814	1.69
fine	2736	+	1407	0.2049	0.81
finer	6452	+	3633	0.1954	0.11
extra fine	18158	+	11063	0.2191	0.06

Summary

- For the same DOF quadratic elements deliver more accurate results then linear elements
- The convergence rate for linear elements in 2D problems is ≈1.2
- For quadratic elements the convergence rate is only slightly increased in comparison to linear elements, and lies significantly below the theoretical value of 2
- In comparison to globally refined meshes adaptive techniques deliver results with same accuracy, but with significantly lower DOF
- Multiple application of adaptive mesh refinement shows reduced improvement with each application
- For the chosen testcases Delaunay meshes do not offer advantages compared to usual COMSOL meshing
- Quality and angle restriction of Delaunay triangulations do not lead to improved results
- Mesh quality optimization is recommended

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