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Energy Transformation Damping

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Abstract: A model for material damping is presented in terms of internal friction and in terms of a variation of stiffness. In the latter case the idea is that the stiffness increases if elastic energy is stored and decreases if elastic energy is released. In case of a single mass spring system “stiffness” refers to the stiffness of the spring; in case of a continuos object “stiffness” refers to the Young’s modules.
The reason for this material damping model is that it simultaneously overcomes problems with linear viscous damping and with Coulomb damping.
The reason for its interpretation as variation of stiffness instead of internal friction is that this substantially simplifies the mathematics.

Keywords: material damping; Coulomb damping; linear viscous damping; non-linear damping; internal friction.

1. Introduction

Typical for real world damping is that the relative energy loss per cycle of a vibrating object, that is the energy loss in a complete cycle divided by the total available energy (elastic+kinetic) at the beginning of the cycle, is roughly frequency and stress-amplitude independent. This holds true for many important structural materials, be it more so with respect to frequency than to amplitude. Many building codes are implicitly based on this principle. As a consequence, if e.g. the amplitude of a vibrating object halves after say 100 cycles, then the amplitude of any other object, made of the same material, also will halve after 100 cycles regardless its frequency or amplitude. Another 100 cycles will halve the amplitude again.

Linear viscous damping (see fig. 1 left) does not reflect this property, as the damping force is proportional with the frequency. *Hysteretic damping* seems to remedy this flaw in case of continuous systems by simply dividing the damping force by the frequency. While this brings the calculations in line with real world damping in case of single mode vibrations, it is unclear how “dividing by frequency” follows from the theory and how it affects linearity or mode coupling.

Also *Coulomb damping* (see fig. 1 mid) does not reflect real world damping, as its relative energy loss per cycle is proportional with the inverse of the amplitude, resulting in amplitude that declines linearly in time. The sound of a Coulomb damped bell would hardly fade away but would simply stop on a certain moment.

To overcome the problems above, another damping model is proposed named *energy transformation damping.* (See fig.1 right)

Section 2 will treat energy transformation damping of a single mass spring system including its duality of internal friction and stiffness variation. Section 3, 4, 5 and 6 will deal with various aspects of energy transformation damping of longitudinally vibrating elastic rods. Section 7 deals with damping of 2D objects with plane stress. Section 8 links stiffness variation with the experiment of A.L Kimball and D.E. Lovell. Section 9 will highlight the problems with Coulomb damping, linear viscous damping and hysteretic damping. Section 10 gives a conclusion.

2. Energy transformation damping for a single mass-spring system

See fig. 1 right. If δ=0 then the hinged bar that connects the mass to a Coulomb friction element is exactly vertical and the force in the spring is zero. If the mass is moving then there is always a friction force F in the bar, with abs(F) is constant. The horizontal component of this force, Fδ/L, is the damping force. The clue is that this force acts in the same direction as the force in the spring if δ(δt) (the product of displacement δ of the mass and its speed δt) is positive and in opposite direction if δ(δt) is negative.

From a *mathematical* point of view we can

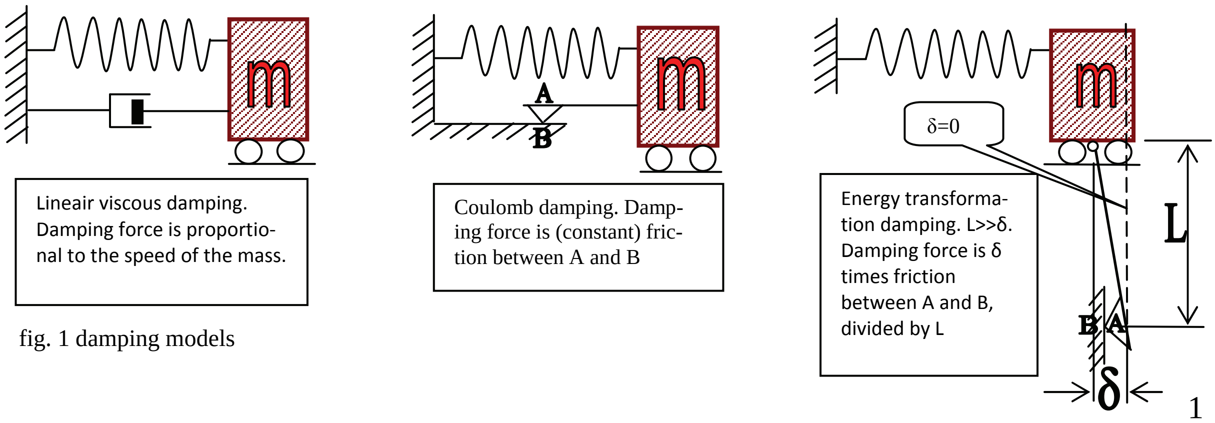

fig. 1 damping models

as well argue that the spring stiffness *k* becomes *k+abs(F/L)* if δ(δt) is positive and *k-abs(F/L)* if δ(δt) is negative or in one formula $k_{dyn(amic)}$= k+abs(F/L)sign(δ(δt)). In a real single mass-spring system abs(F/L) is a property of the spring since F and L do not exist separately. Substituting α for abs(F/L)/k results in k_{dyn}=k(1+αsign(δ(δt))). That means that if δ(δt) is *positive*, that is if elastic energy is *stored*, the stiffness is k(1+α). If δ(δt) is *negative*, that is if elastic energy is *released*, the stiffness is k(1-α).

Because α is now the relative variation of the spring stiffness, with in general α<<1, it is also the relative energy loss for any amount of energy that is transformed from kinetic into elastic energy or vise versa. The equation of a single mass-spring system with energy transformation damping simply is:

$$m\delta tt+k_{dyn}\delta=0.$$

It is interesting to note that in contrast with hysteretic damping, no reference is made to any harmonic vibration; the expression for the spring stiffness simply holds true for any force-strain path. Moreover the loss of energy is proportional to transformation of elastic energy into kinetic energy and vise versa, explaining the name of this damping model. Finally as a result: in case of a harmonic vibration the relative energy loss of one complete cycle is approximately 4α since all energy is 4 times transformed and is as such frequency and amplitude independent.

3. Equation of an elastic rod with energy transformation damping

Although it is possible to extend the above mentioned approach of material damping to 2D and 3D cases the focus will be on the 1D case of an elastic rod to make the underlying mechanism more transparent.

Replacing the spring stiffness by the Young's modules E, δ by ux and with α the relative variation of E we get:

$E_{dyn}=E(1+\alpha sign(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}))$ and with the switch function $SWI=sign(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t})$, $E_{dyn}=E(1+\alpha SWI)$

Let u(x,t) be the displacement of a longitudinally freely vibrating elastic rod without damping with m as distributed mass, A as constant cross-section and the Young's modulus E a function of x, then:

$$m\frac{\partial^2 u}{\partial t^2}-A\frac{\partial E}{\partial x}\frac{\partial u}{\partial x}-AE\frac{\partial^2 u}{\partial x^2}=0 \qquad (1)$$

Substituting E_{dyn} for E, we find the equation of the longitudinally vibrating elastic rod with energy transformation damping.

$$m\frac{\partial^2 u}{\partial t^2}-AE(\alpha\frac{\partial SWI}{\partial x}\frac{\partial u}{\partial x}+\alpha SWI\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2})=0 \quad (2)$$

4. Single mode vibration

Fig. 2 is an extruded plot of SWI viewed in z-direction of a vibrating rod with length=pi, initial condition $u(t_0)$=sin(3x), boundary $u(x_0)$=0, $u(x_{pi})$=0, mass=1 and A=1, E=1.

Note: horizontal displacements of sin(3x) are of course unrealistic big if the length of the rod is only pi, but geometrical linearity is assumed throughout this article.

The initial condition results in a single mode vibration and is as such a special case. As can be observed SWI is a function of time only, since the blue and red stripes are parallel to the x-axis. SWI is either –1 or +1 over the entire length of the rod and for that reason not a function of x, so SWIx=0.

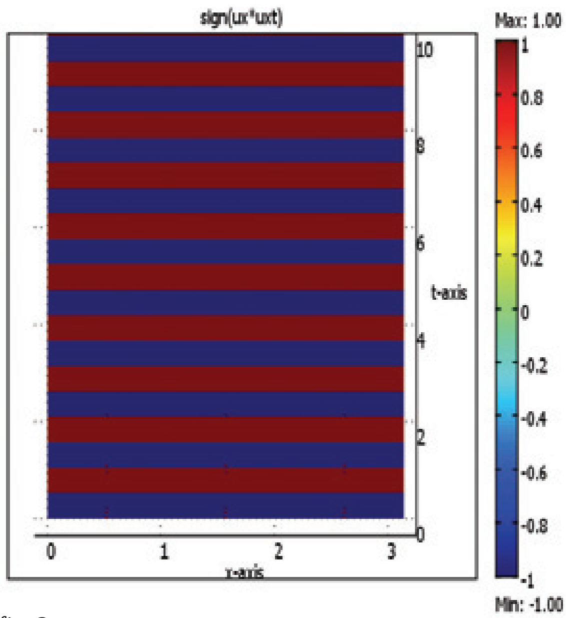

fig. 2

Comsol MPH can now directly solve (2). As could be expected, *the energy loss per cycle* turned out to be indeed approximately 4α times the available energy because in a complete cycle all energy is 2 times transformed from kinetic to elastic and 2 times from elastic to kinetic. So, as with the single mass-spring system, it is both frequency *and* amplitude independent.

5. Multi mode vibration

Fig. 3 is an extruded plot of SWI viewed in z-direction of the same rod as in fig. 2 but now with initial conditions $u(t_0)$=sin(x)+sin(2x) resulting in a multi mode vibration. Now clearly SWI is a function of both time and x and jumps at several locations in the x-domain from +1 to –1 and from –1 to +1. Every horizontal line represents a point of time and each crossing of such a line from red to blue or from blue to red marks a jump of SWI.

Here non-linearity kicks in because SWIx represents *mode coupling*. It simply does not exist with initial conditions $u(t_0)=\sin(x)$ or $u(t_0)=\sin(2x)$ but it pops inevitably up for the initial condition $u(t_0)=\sin(x)+\sin(2x)$. SWIx has the value +2/-2 at the transition blue-red respectively red-blue and is everywhere else zero. To solve (2) we replace “sign” by its smoothed version “flsmsing”.

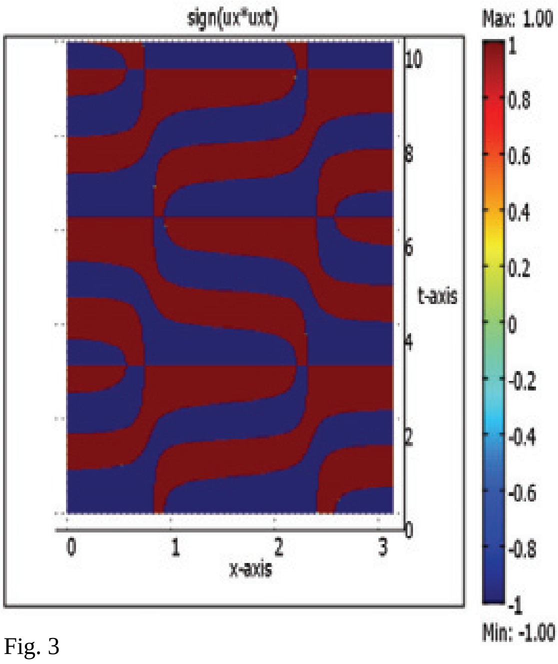

Fig. 3

Also pre-stressing of the rod always results in non-linearity even in case of a single mode vibration. Consider for example a rod with length pi, initial condition $u(t_0)=x+\sin(x)$, boundary conditions $u(x_0)=0$, $u(x_{pi})=pi$, mass=1 and AE=1. This rod is pre-stressed because of the linear part of the initial condition in combination with the second boundary condition. In this case always an alternating blue-red / red-blue transition exists in the middle of the rod and hence SWIx pops up despite the fact that it is zero for an initial condition $u(t_0)=x$ or $u(t_0)=\sin(x)$. The most important difference between the rod of section 4 and the pre-stressed rod is that the former releases or stores elastic energy over its full length where the latter releases elastic energy in one half while storing it in the other half. The released and the stored amount of energy is not equal however; the balance transforms to kinetic energy.

So obviously in case of multi mode vibration and or pre-stressing, not all energy is transformed from kinetic to elastic and vise versa. Part of the elastic energy is relocated from one spot of the vibrating object to the other and the rest of the energy is transformed. Relocating also means storing and releasing and hence energy loss. For that reason the name *energy transformation damping* is only 100% adequate in case of an object with a single mode vibration without eigen stresses. The bottom line however is that equation (2) automatically keeps track of all these phenomena so there is not too much reason to worry about the name of the damping model. Just relax and let Intel and Comsol do the hard part of the job.

6. Results

Implementing equation (2) for the rod of section 5 with COMSOL Multiphysics is remarkable simple.

Select 1D; PDE Modes; PDE Coefficient Form. Make global expressions SWI=flsmsign(ux*uxt), A=1, E=1, m=1, α=0.1 (α is too high but could be reasonable for soil mechanics).

Subdomain Settings: fill in for c AE(1+αSWI) and for e_a m. All other symbols are 0. The equation is now interpreted as:

$$m(\partial^2 u/\partial t^2)- \partial\{AE(1+\alpha SWI)(\partial u/\partial x)\}/\partial x=0$$

and this is indeed the counterpart of (2).

Length of the rod is pi with 30 elements Lagrange Quadratic. Init $u(t_0)=\sin(x)+\sin(2^*x)$. Boundary $u(x_0)=0$ $u(x_{pi})=0$

Solver parameters: generalized alpha, max time step 0.001. Accept the default values for all other input.

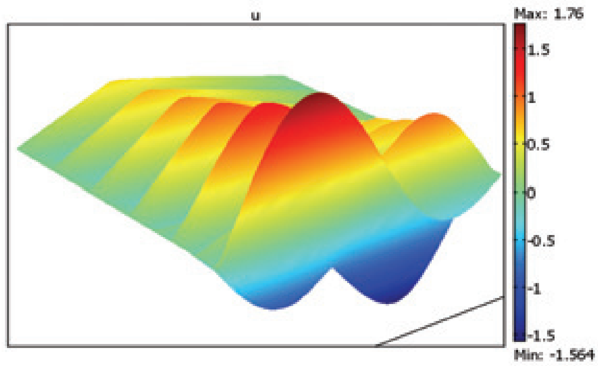

fig. 4 solution of equation (2)

See for solution u figure 4.

Obviously Comsol MPH can solve (2). The vertical axis shows the horizontal displacements of material points. The evolution of the solution u differs substantially from similar solutions with Coulomb damping, linear viscous damping and hysteretic damping as given in section 9.

Fig. 5 is a plot of the same case as fig. 4 however now the rod is pre-stressed with ux=0.5. Displacements u=0.5x due to pre-stressing were subtracted from the solution to make figure 4 and 5 comparable. As can be observed pre-stressing increases the damping. Moreover the vibration stops

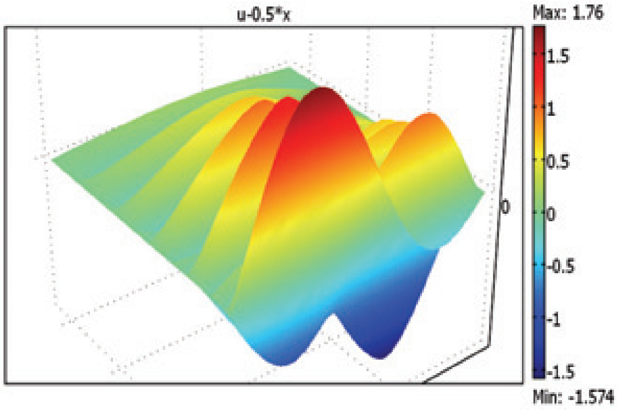

fig. 5 solution (2) for pre-stressed rod

on a certain moment rather than that it fades away as in the non pre-stressed case of fig. 4. In fact if the pre-stress is much higher that the stress variation caused by the vibration we end up with Coulomb damping. (See also section 9)

7. Plane stress

As could be expected energy transformation damping is also applicable to higher dimensions such as plane stress. We can directly build it in in the structural mechanics mode. A vector product replaces ux(uxt) as argument for the sign function. In case of plane stress elastic energy is stored if the expression below is positive and released if negative.

$$\begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \frac{1-\gamma}{2} \end{bmatrix} \begin{pmatrix} ux \\ vy \\ uy+vx \end{pmatrix} \bullet \begin{pmatrix} uxt \\ vyt \\ uyt+vxt \end{pmatrix}^T$$

u is displacement in x direction
v is displacement in y direction
γ is Poisson modules

Suppose γ=0.3 than make global expression In=(ux+.3vy)uxt+(.3ux+vy)vyt+.35(uy+vx)(uyt+vyt)
Area lx=π ly=π/2,
Mapped mesh, mesh size normal 8x15 elements, Lagrange Quadratic.
Subdomain settings:
E=1+0.1*flsmsign(In), γ=0.3, ρ=1, thickness=1.
Init $u(t_0)$=(sin(x)+sin(2*x))*sin(2*y) All boundaries Rx=0 Ry=0

Tuning the solver parameters is quite a challenge; the following parameters gave a convincing result after 20 minutes' run on a dual core 3 GB laptop using version 3.5a:
Solver parameters:
Range (0,0.2,30), Generalized alpha, Specified times, Free. Init step 0.0001, max step auto, time step increase delay 50, amplitude factor 1, Jacobian update on every iteration. All others default.

8. Energy transformation damping and the experiment of Kimball and Lovell

In 1927 A.L. Kimball and D.E. Lovell carried out an experiment leading to the understanding that relative energy loss per stress cycle is independent of the rate of change of the stresses and practically independent of the stress amplitude.

(See fig. 6) The bar to be tested is supported by two bearings and connected to an engine by means of a flexible coupling and carries an adjustable load at its cantilevering end. As long as the engine is switched off it is self evident that the bar bends in a vertical plane but as soon as the engine is switched on, the plane of bending tilts over an angle Φ around the axis trough the two bearings. At

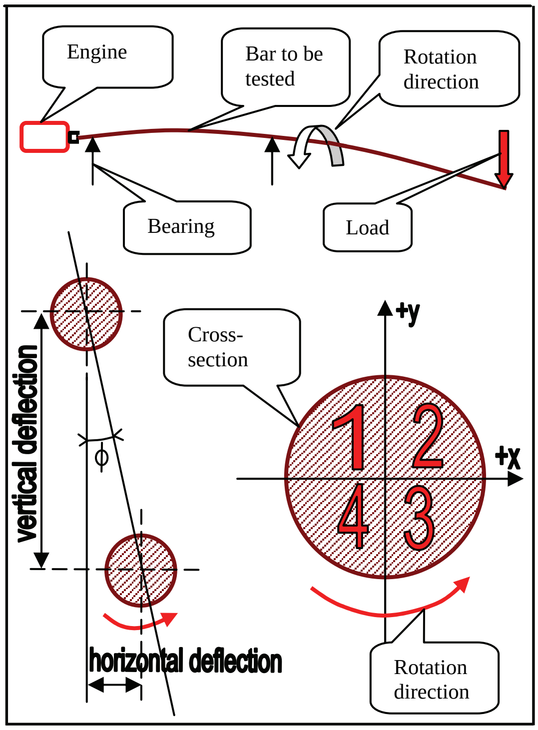

fig. 6 experiment of Kimball and Lovell

the time it was very surprising that Φ turned out to be fully independent of the speed of the engine and to a great extent independent of the load as long as the load was not too big. The conclusions of the authors, although otherwise formulated, boiled down to this:

If Δ is the relative amplitude decay of a vibrating object made of the same material as the rotating bar, then Φ=Δ/π. (The relative amplitude decay is the amplitude decay over one cycle divided by the amplitude at the beginning of the cycle.)

The question is now: does the approach with the dynamic Young's modules lead to the same conclusion? The answer is yes.

Consider the cross-section of the bar as given in fig. 6. Because of the rotation of the bar, the absolute value of the stress in quadrant 1 and 3 is decreasing since all material points of these quadrants move toward the neutral axis. Similarly the absolute value of the stress of quadrant 2 and 4 is increasing. For that reason the dynamic Young's modules of quadrant 1 and 3 is E(1-α) and of quadrant 2 and 4 E(1+α). While the moment mx acts around the x-axis only, bending is around the x-axis and the y-axis according to the following equation:

$$E\begin{bmatrix} Ixx & Ixy \\ Iyx & Iyy \end{bmatrix}\begin{pmatrix} \kappa x \\ \kappa y \end{pmatrix} = \begin{pmatrix} mx \\ 0 \end{pmatrix} \quad (3)$$

- Ixx=Iyy is simply the inertia moment of a circle=$\pi R^4/4$, since the effects of E_{dyn} cancel out.
- Ixy=Iyx is 4α times the inertia product of a quarter circle with respect to the x- and y-axis=$\alpha R^4/2$, because all quadrants add up since quadrant 1 and 3 with –α have also a negative xy product.
- Kx and κy are the curvatures around x-axis and y-axis respectively.

Now with (3) we can calculate κy/κx as $(\alpha R^4/2)/(\pi R^4/4)=2\alpha/\pi$. So κy/κx is obviously constant for each section of the bar, implicating that the plane of bending of the bar tilts over a angle Φ=2α/π. However since 4α is the relative energy loss of one cycle, the relative amplitude decay is 2α and hence 2α=Δ and Φ=Δ/π. QED

9. Alternative damping models

It is interesting to see where coulomb damping, linear viscous damping and hysteretic damping fail if applied to a simple continuous system as a rod.

9.1 Coulomb damping

Let F be the force in a longitudinally vibrating elastic rod with Coulomb damping and β a material-damping constant, then:

F=AE∂u/∂x+βAEsign($\partial^2 u/\partial x\partial t$) and the equation of the rod: m$\partial^2 u/dt^2$-∂F/∂x=0
or with SWI=sign($\partial^2 u/\partial x\partial t$):

$$m\frac{\partial^2 u}{\partial t^2} - AE(\beta\frac{\partial SWI}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}) = 0 \quad (4)$$

See for a solution of (4) the extruded plot fig. 7. The boundary and initial conditions are the same of those of fig. 4 and β=0.1. Only the type of damping is different. The subdomain settings are: c=1, a_e=1 and γ=-0.1*flsmsingn(uxt,0.01).

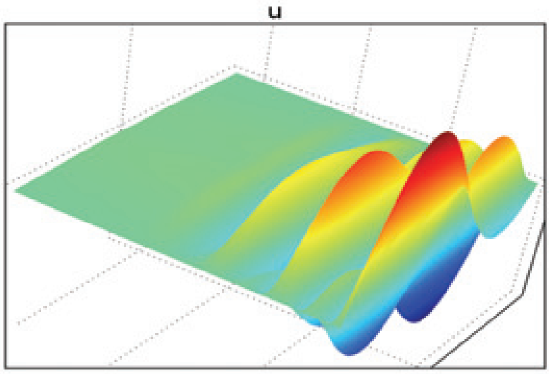

Fig. 7 Coulomb damping

It appears that the amplitude declines linearly in time. For that reason we conclude that Coulomb damping overestimates (underestimates) damping of amplitudes smaller (bigger) than the amplitude for which β holds true.

Typical for Coulomb Damping is that after vibrating a small deformation remains. This is also the case with energy transformation damping in combination with pre-stressing. Coulomb damping is non linear.

9.2 Linear viscous damping

Let F be the force in a longitudinally vibrating elastic rod with linear viscous damping and γ a material-damping constant, then:
F=AE∂u/∂x+γAE$\partial^2 u/\partial x\partial t$ and the equation of the rod:

$$m\frac{\partial^2 u}{\partial t^2} - AE(\gamma\frac{\partial^3 u}{\partial x^2\partial t} + \frac{\partial^2 u}{\partial x^2}) = 0 \quad (5)$$

See for a solution of (5) the extruded plot of fig. 8.
E=1, A=1, l=pi, m=1, γ=0.1

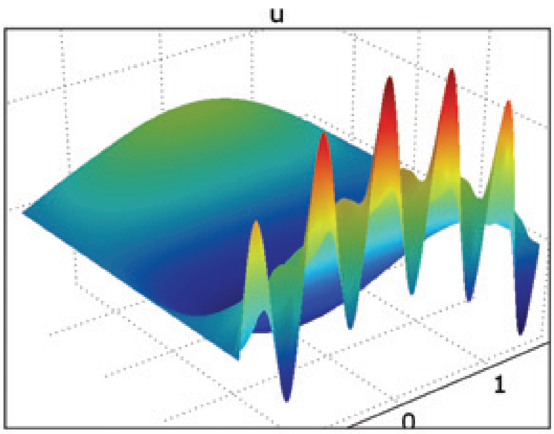

Fig. 8 linear viscous damping

Time-domain is 0-7.3 sec, initial condition is: $u(t_0)$=sin(x)+sin(10x). The subdomain settings are c=1, a_e=1, gamma=-0.1*uxt.

It appears that the amplitude decay of sin(10x) is very big. In fact its damping is nearly critical while damping of sin(x) is far from critical. For that reason we conclude that linear viscous damping overestimates (underestimates) damping of frequencies higher (lower) than the frequency for which γ holds true.

In a universe with linear viscous damping all musical instruments have the same timbre if we are at least lucky enough that they are not over critically damped or hardly damped at all!

9.3 Hysteretic damping

The equation of a longitudinally vibrating elastic rod with hysteretic damping is:

$$m\frac{\partial^2 u}{\partial t^2} - AE\left(\frac{\gamma}{\omega_i}\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^2 u}{\partial x^2}\right) = 0 \qquad (6)$$

The idea behind hysteretic damping is that each principle mode gets its own damping characteristic.

Take a rod with length pi and suppose the damping constant γ/ω_1 for the first principle mode sin(x) is 0.1, then the damping constant γ/ω_2 for the second principle mode sin(2x) is 0.1/2=0.05 because $\omega_2/\omega_1=2$. For the third principle mode sin(3x) it is 0.0333 and so on.

This will certainly result in the same relative amplitude decay for sin(x) as for sin(10x) in fig. 8 and is as such in line with the Kimball Lovell experiment. In case of linear viscous damping the energy that is dissipated at a certain location per unit of volume is γEuxt (e.g. Watt/m^3). However the problem with hysteretic damping is that there is no such unique relation between energy dissipation and uxt as can be concluded from the following example (see fig. 9):

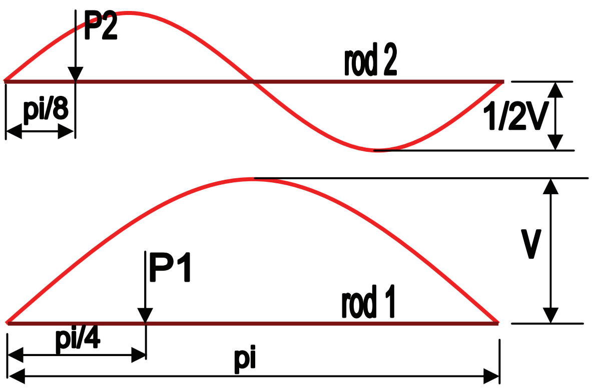

Fig. 9 Two velocity distributions

Consider two identical rods with length pi and ut=Vsin(x) for rod 1 and ut=½Vsin(2x) for rod 2. P1 and P2 are points located such that the slope of the red lines is the same, e.g. at distances pi/4 and pi/8 respectively. Since the slopes are the same, also uxt is the same for P1 and P2, however the energy dissipation in case of hysteretic damping at P1 is 2 times bigger than at P2 because $\omega_2=2\omega_1$. This indeed proves that in case of hysteretic damping there is not an unambiguous correlation between energy dissipation and uxt.

But if not, how can a material particle "be aware" of principle modes and "know" what ω_i to choose? The theory of hysteretic damping does not give a clue anyway and the best thing we can hope for is that it is correct but incomplete. However, in contrast with energy transformation damping, the nature of non-linearity or mode coupling remains fuzzy.

10. Conclusion

Damping is non linear and should not be squeezed into a linear model. Coulomb damping is a non-linear damping model but its basis is wrong. The issue is not that increasing or decreasing of the *strain* controls the direction of the *friction*. The issue is that increasing or decreasing of the *absolute value of the strain* controls the *stiffness*. Exactly *that* is the basis of energy transformation damping.

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