Presented at the COMSOL Conference 2010 Paris

COMSOL Conference 2010, Nov. 17-19, Paris

Convergence Rates for Models with Combined 1D/2D Subdomains

Bradji Abdallah

University of Badji Mokhtar Annaba (ALGERIA)

Holzbecher Ekkehard

Litz Marie-Sophie

Georg-August-University Göttingen (GERMANY)

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

Introduction

- Numerical models deliver an approximation u_h for the ‚real' solution u, (only)
- The numerical solution depends on the grid spacing h (and eventually other numerical parameters, for example the timestep)
- The 'error' can be measured as

$$\|u - u_h\|$$

with the norm $\| \; \|$ as a distance measure

(Analytical) Solutions & Norms

In case to compute the error we need to decide about *u* and ‖ ‖ .

- For ‚simple‘ testcases an analytical solution exists and can be computed easily.
- If no analytical solution is known we may take a high precision numerical solution as a substitute.
- Concerning the norm, the most frequent choices are
  - maximum norm
  - average norm
  - energy norm

$$\|e\|_{\infty} = \max\left(\left|u_h - u\right|\right)$$

$$\|e\|_{2,0} = \sqrt{\int_{\Omega} \left(u_h - u\right)^2}$$

$$\|e\|_{2,1} = \sqrt{\int_{\Omega} \left(u_h - u\right)^2 + \int_{\Omega} \left(\partial u_h - \partial u\right)^2}$$

Convergence Rates

In case of convergence of the numerical solutions we have

$$\left\| u - u_h \right\| \to 0 \quad \text{for} \quad h \to 0$$

If we assume the following approximation for the error

$$\left\| u - u_h \right\| \approx Const \cdot h^{\vartheta}$$
$$= O\left(h^{\vartheta} \right)$$

we obtain the convergence rate ϑ as a measure for the convergence. The higher the convergence rate, the faster the convergence for $h \to 0$.

$$\left.\begin{aligned} \vartheta &= 1, \text{ linear convergence, } \frac{\left\| u - u_h \right\|}{\left\| u - u_{h_0} \right\|} = 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots \\ \vartheta &= 2, \text{ quadr. convergence, } \frac{\left\| u - u_h \right\|}{\left\| u - u_{h_0} \right\|} = 1, \tfrac{1}{4}, \tfrac{1}{16}, \tfrac{1}{8}, \ldots \end{aligned}\right\} \text{for } h = h_0, \frac{h_0}{2}, \frac{h_0}{4}, \ldots$$

Irregular Meshes & DOFs

- If we compute the error for two different mesh sizes, we can obtain the convergence rate by:

$$\vartheta = \frac{\ln\left(\left\|u - u_{h_1}\right\|\right) - \ln\left(\left\|u - u_{h_2}\right\|\right)}{\ln\left(h_1\right) - \ln\left(h_2\right)}$$

- For irregular meshes there is no single mesh constant *h.* Instead we may use the number of degrees of freedom (DOF) as measure of the mesh refinement. Then we obtain ϑ from

$$\vartheta = -2\frac{\ln\left(\left\|u - u_{h_1}\right\|\right) - \ln\left(\left\|u - u_{h_2}\right\|\right)}{\ln\left(DOF_1\right) - \ln\left(DOF_2\right)}$$

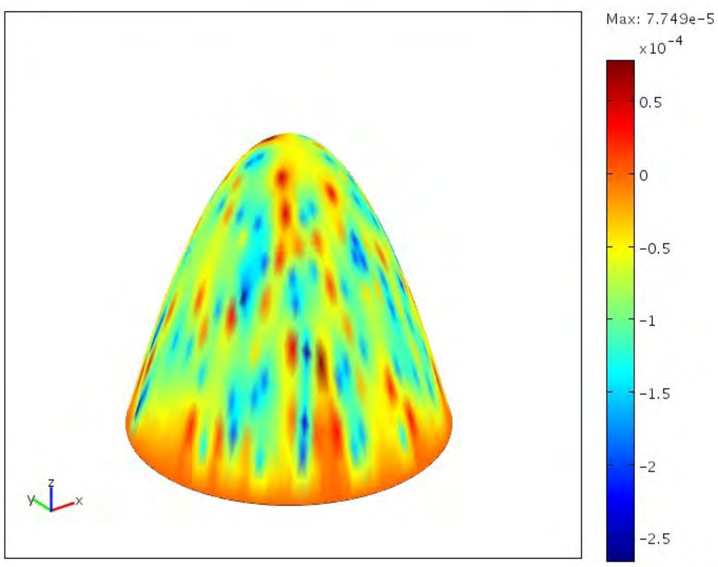
for 2D
Jänicke & Kost (1999)

Example 1:
Potential Eq. & Dirichlet Conditions

$$-\nabla^2 u = 1$$

$$u(x,y) = -(x^2 + y^2 - 1)/4$$

DOF	$\|u - u_h\|$	ϑ	
1561	**0.0017**	**1.95**	
6145	**0.00044**		**2.00**
24383	**0.00011**	**1.98**	
97163	**0.000028**		



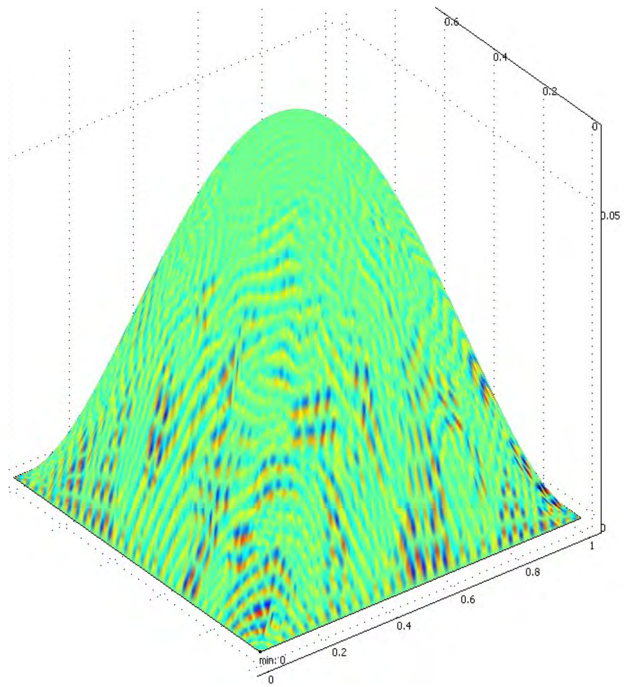
Bradji & Holzbecher (2007,2008)

Element order	Norm	Conv. rate
1	average	2
1	energy	1
2	average	3
2	energy	2
2	maximum	3

Example 2: Potential Eq. & Dirichlet Conditions

$$-\nabla^2 u = 1$$

$$u(x,y) = \sin(xy)\sin\big((1-x)(1-y)\big)$$



Bradji & Holzbecher (2007, 2008)

DOF	$\|u-u_h\|_1$	ϑ_1		$\|u-u_h\|_2$	ϑ_2	
520	$4.1\ 10^{-4}$	1.81				
2017	$1.2\ 10^{-4}$		1.97	$9.4\ 10^{-6}$		3.00
7945	$3.1\ 10^{-5}$	1.97		$1.2\ 10^{-6}$	2.98	
31537	$8.0\ 10^{-6}$		2.01	$1.5\ 10^{-7}$		3.11
$1.25\ 10^5$	$2.0\ 10^{-6}$	2.00		$1.8\ 10^{-8}$	2.97	
$5\ 10^5$	$5.0\ 10^{-7}$			$2.3\ 10^{-9}$		

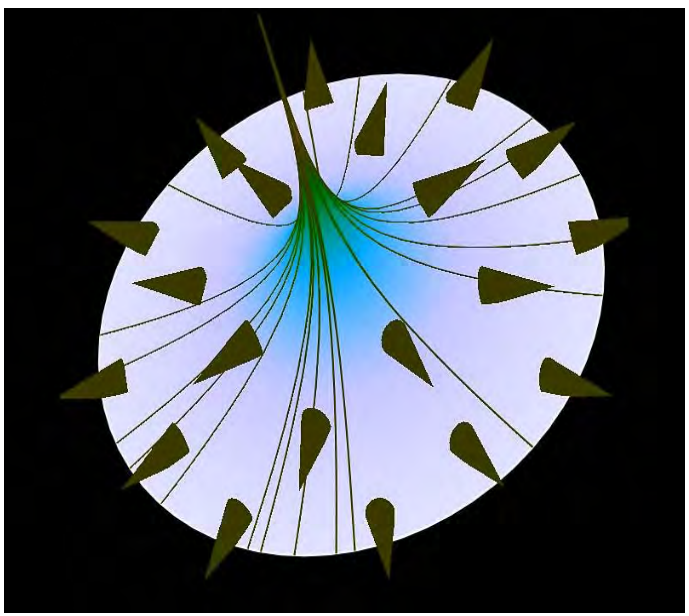
Details for average norm

Element order	Norm	Conv. rate
1	average	2
1	energy	1
1	maximum	3
2	average	3
2	energy	2
2	maximum	3

Example 3: Poisson Equation with Dirac Right Hand Side

$$-\nabla^2 u = \delta(0)$$

$$u(x,y) = -\ln(r)/2\pi = -\ln(r^2)/4\pi$$



Bradji & Holzbecher (2008)

First order elements

DOF	time	$\|u-u_h\|$	ϑ_1	
777	0.046-0.06	0.9784	1.97	
3041	0.125	0.2557		1.90
12033	0.5-0.547	0.0692	1.69	
47873	2.344	0.0214		1.37
190977	18.11	0.0083		

Second order elements

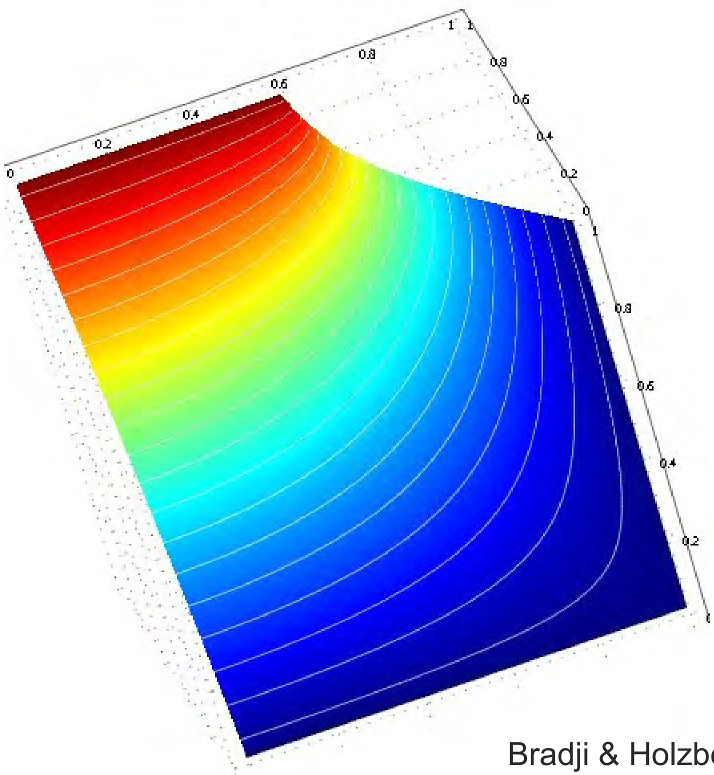
DOF	time	$\|u-u_h\|$
3041	0.141	0.0601
12033	0.515	0.0277
47873	2.563	0.0138
190977	13.95	0.0069

Details for average norm

Example 4: Potential Eq. with Dirichlet- and Neumann conditions

$\Delta u=0$

Analytical solution from
Schwarz-Christoffel Toolbox



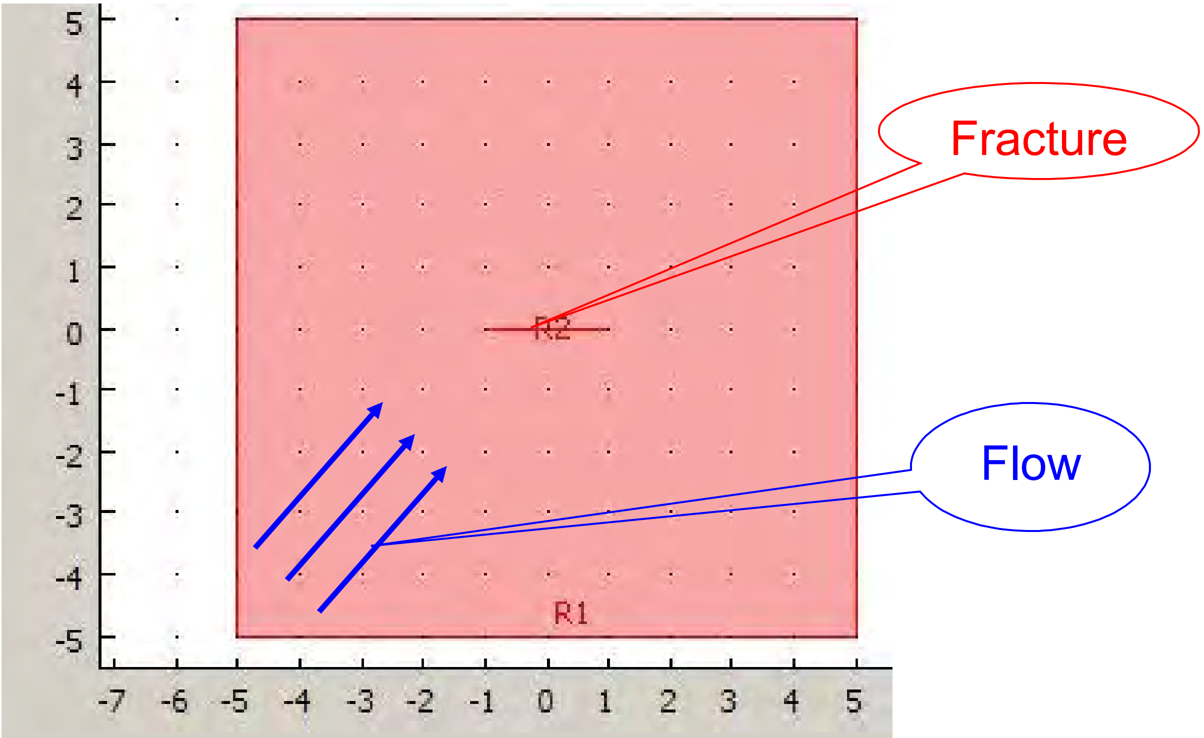
DOF	# lin. elements	$\|u - u_h\|$	ϑ	
527	992	0.249	0.95	
2045	3968	0.129		1.07
8057	15872	0.0614	1.05	
31985	63488	0.0297		1.02
127457	253952	0.0146		

DOF	# quad. elem.	$\|u - u_h\|$	ϑ	
2045	992	0.1366	0.96	
8045	3968	0.0702		1.08
31985	15872	0.0332	1.23	
127457	63488	0.0142		

Details for average norm

Bradji & Holzbecher (2007)

Combined 1D/2D: Set-up 1



Thin fracture in a constant flow field

Mathematical approach: Darcy‘s Law in Fracture and Matrix

Differential Equations & Analytical Solution

Matrix (2D): $\nabla K_{low} \nabla \varphi = 0$

- low hydraulic conductivity

Fracture(1D): $\nabla K_{high} \nabla \varphi = 0$

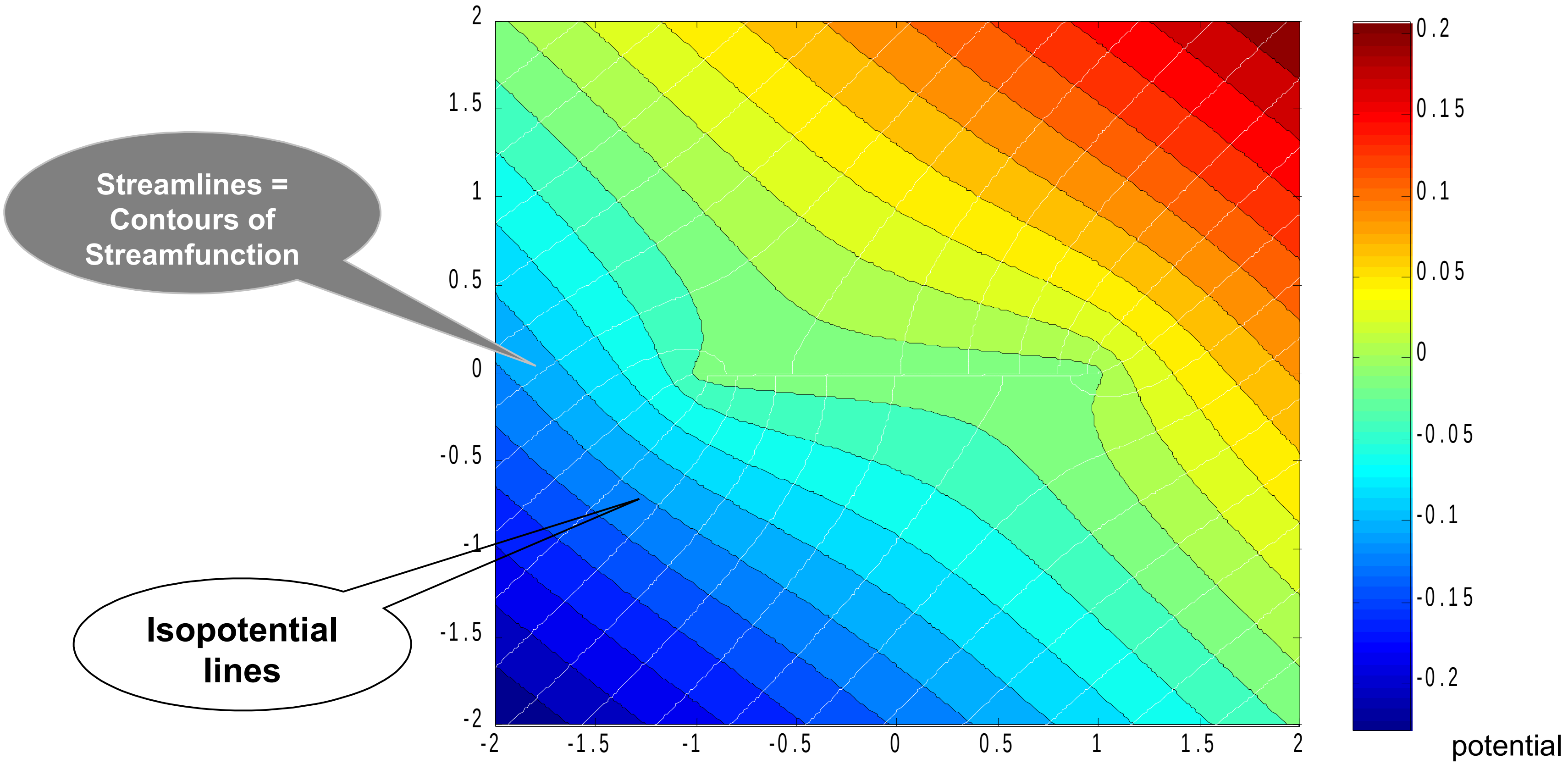
- high hydraulic conductivity

Analytical solution (complex potential):

$$\overline{\Phi}(z) = -i\Phi_o\left(z\cos(\alpha) - i\sqrt{z^2 - a^2}\sin(\alpha)\right)$$

For more details see Holzbecher et al. 2010, this conference

MATLAB Visualization



Analytical solution for real and imaginary part

Numerical Solution*

* for real potential part only

2D Geometry

- total domain: diffusion equation for real potential φ
- boundary conditions: Dirichlet

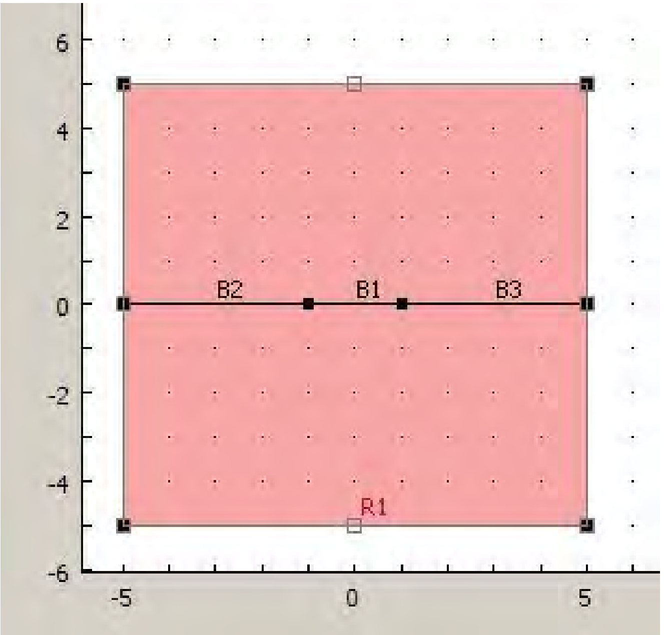
1D Geometry (for lower dimensional case)

- diffusion equation for real potential φ
- boundary conditions: Neumann

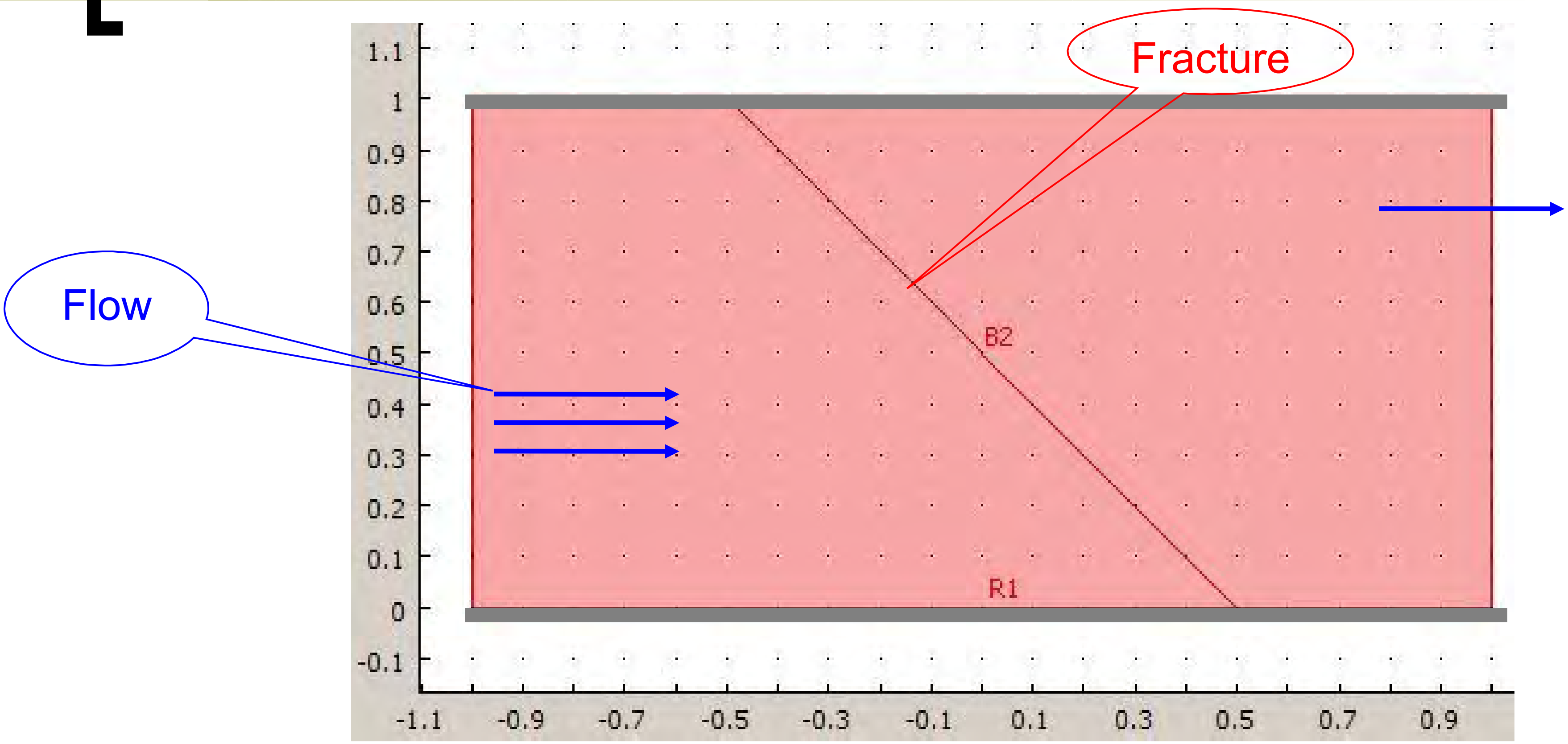
Coupling:

- solutions identical at fracture (B1)

Coupling is introduced using subdomain extrusion variable from 1D to 2D



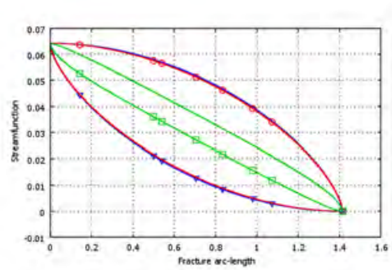
Combined 1D/2D: Set-up 2



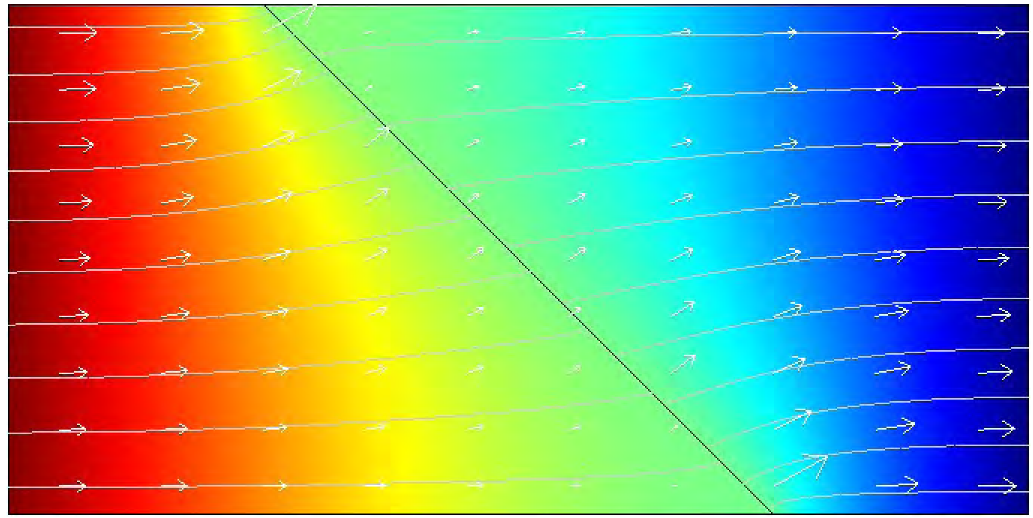
Potential equations in 1D (B2) and 2D (R1)
Bounsary conditions: Dirichlet and Neumann

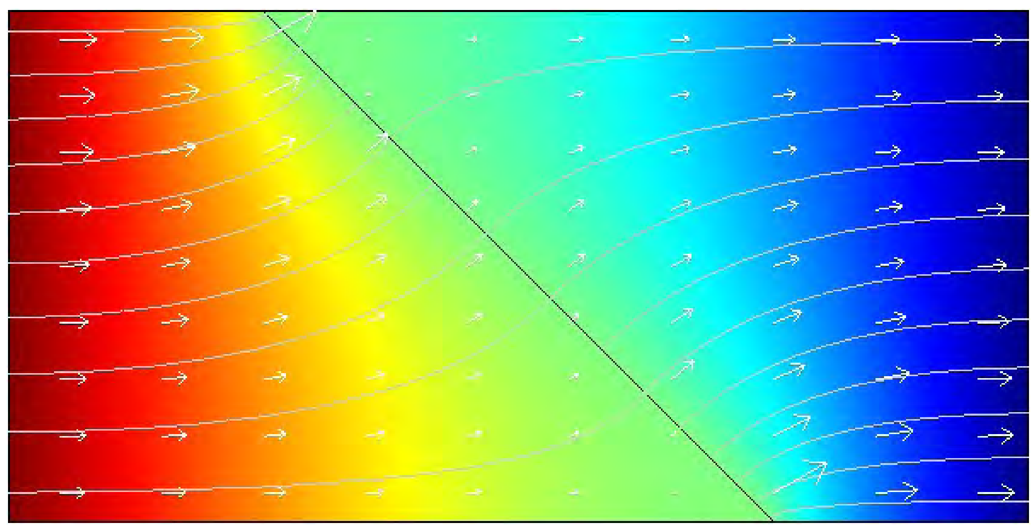
Flow Pattern; Variation of K_{ratio}

Angle: 45°
Width: 0.01
K_{ratio} : 100 (top)
and 10000 (bottom)



1D lower-dimensional fracture





Model Runs

2^{nd} order elem.; max-norm; set-up 1

DOF	$\|e\| \cdot 10^2$	ϑ	
2577	9.2220	0.5039	
9999	6.5535		0.5090
39387	4.6231	0.5398	
156339	3.1867		0.5684
622947	2.1513	0.5345	
2486979	1.4861		

2^{nd} order elem.; average-norm, set-up 1

DOF	$\|e\| \cdot 10^2$	ϑ	
2577	5.5346	1.0643	
9999	2.6900		1.0421
39387	1.3168	1.0418	
156339	0.6422		1.0639
622947	0.3078	1.1243	
2486979	0.1414		

$$\|e\| = \|u_h - u\|$$

Similarly for energy norm, second order elements and set-up 2.
For all details see paper!

Conv. Rates 2D/1D Combined Models

Set-up 1

	1st order elements	2nd order elements
Maximum norm	0.5	0.5
Average (L2) norm	1.0	1.0
Energy norm	0.5	0.5

Set-up 2

	1st order elements	2nd order elements
Maximum norm	0.7	0.7
Average (L2) norm	1.0	1.0
Energy norm	0.5	0.5

Convergence rates turn out to be very low and independent of element order

Comparison with Pure 2D

Convergence rate for set-up 1 with full 2D approach for fracture

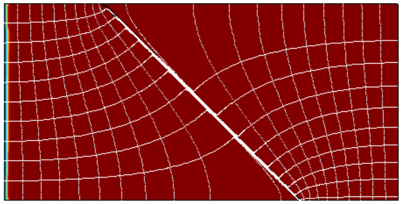
	1st order elements	2nd order elements
Maximum norm	0.72	0.78
Average norm	1.78	1.73
Energy norm	1.20	0.75

Convergence rates for the pure 2D approach are in all cases (concerning norms and element order) higher than for the coupled 2D/1D approach.

However for the given set-ups the convergence rates for the 2D approach are much smaller than for the single dimensional examples, seen before.

Conclusions

The convergence rates for the combined 1D/2D model are significantly reduced in comparison to the pure 2D-set-up and even more when compared with single dimensional examples. Moreover there seems to be no dependence of the finite element order. This is a clear indication that the finite element discretization is not the crucial task of the numerical solution. The coupling between the 1D and 2D domains most likely is the limiting process in the ent constellation - with the substantial nega on the convergence.



Merci beaucoup