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Semismooth Newton Method for Gradient Constrained Minimization Problem : Using of COMSOL Multiphysics

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Abstract: We treat a gradient constrained minimization problem which has applications in mechanics and superconductivity. A particular case of this problem is the elastoplastic torsion problem. In order to solve the problem we developed an algorithm in an infinite dimensional space framework using the concept of the generalized Newton derivative. The Desktop environment of COMSOL Multiphysics 4.1 was used to test the algorithm with Argyris finite elements. Numerical tests demonstrate mesh-independent behaviour of the method.

Keywords: optimization with gradient constraint, variational inequality, semismooth Newton method, Argyris finite element, COMSOL Multiphysics 4.1

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be simply connected bounded Lipschitz domain. The set $K = \{v \in H_0^1(\Omega) | \quad |\nabla v| \leq 1 \text{ a.e. in } \Omega\}$ is non-empty, convex and closed in $H_0^1(\Omega)$. For a given $f \in H^{-1}(\Omega)$ we treat the variational inequality : to find a solution $u \in K$ such that

$$\int_\Omega \nabla u \nabla(v-u)dx \geq \langle f, v\rangle \quad \forall v \in K. \qquad (1)$$

This variational inequality can equivalently be formulated as a gradient constrained minimization problem: to find a solution $u \in K$ such that

$$J(u) = \min_{v\in K} J(v)$$

where

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx. \qquad (2)$$

The cost functional $J(v)$ is lower semicontinuous, strictly convex and proper. Thus, we can claim the existence of unique solution $u \in H_0^1(\Omega)$ by using of the standard arguments for constrained minimization problems (e.g. [8]).

In [4] it is shown that even under smooth data assumption the solution cannot have the regularity higher than $W^{2,p}(\Omega), 1 < p < \infty$.

1.1 Optimality system

The existence of Lagrange multipliers $\lambda \in L^\infty(\Omega)$ for the case with $f = d > 0$ constant was proven in [1]. For the general case $f \in L^2(\Omega)$ the question on the regularity of Lagrange multipliers is still open. Let us start with the formal Lagrangian for the problem :

$$\mathscr{L}(v,\lambda) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx + \\ + \int_\Omega \lambda(|\nabla v|^2 - 1)dx.$$

Now if u^* denotes a solution (the existence of which we know) then the formal Karush-Kuhn-Tucker (KKT) conditions for a Lagrange multiplier λ^* is:

$$\int_\Omega (1+2\lambda^*)\nabla u^* \nabla v dx = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega)$$
$$\lambda^* \geq 0, \quad |\nabla u^*|^2 - 1 \leq 0, \quad (\lambda^*, |\nabla u^*|^2 - 1) = 0 \qquad (3)$$

This system has a nonlinear structure and we want to use the Newton method for solving it. Since with the last row it is impossible to apply Newton method we reformulate the optimality system in the following way:

$$(\nabla u^*, \nabla v) + 2(\lambda^*, \nabla u^* \cdot \nabla v) = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega),$$
$$\lambda^* = \max(0, \lambda^* + c(|\nabla u^*|^2 - 1)) \qquad (4)$$

where $c > 0$ is fixed and the max-operation is defined pointwise.

1.2 Regularization

Further we continue with the infinite dimensional form of the system to solve. The Newton method in Banach space requires the Frechet differentiability of the operator describing the equation. Since the pointwise max-function is not Frechet differentiable we apply semismooth Newton method where the Frechet derivative is replaced by a generalized, so called, Newton derivative.

We recall the definition of the Newton derivative. Let $F : D \subset X \to Y$ be a continuous mapping, X and Y are Banach spaces, and D is an open domain in X. We consider nonlinear operator equation:

$$F(u) = 0, \tag{5}$$

Definition 1 *[6] The mapping $F : D \subset X \to Z$ is called generalized differentiable (Newton differentiable) on the open subset $U \subset D$ if there exists a family of generalized derivatives $G : U \to L(X,Z)$ such that*

$$\lim_{\|h\|\to 0} \frac{1}{\|h\|} \|F(x+h) - F(x) - G(x+h)h\| = 0,$$

for every $x \in U$.

Now we consider Newton differentiability of the pointwise max-operator. For $\delta \in R$ we introduce the following candidate for its generalized derivative of the form:

$$G_\delta(u)(x) = \begin{cases} 1 & \text{if } u(x) > 0 \\ \delta & \text{if } u(x) = 0 \\ 0 & \text{if } u(x) < 0. \end{cases}$$

Proposition 2 *[3] The mapping $\max(0,\cdot) : L^q(\Omega) \to L^r(\Omega)$ with $1 \le r < q \le \infty$ is Newton differentiable on L^q and G_δ is a generalized derivative.*

To use the semismooth Newton method we have to regularize the equation (4) first.

It is clear that for the solution u^* the complementarity condition in (3) is valid and

$$\int_\Omega \lambda^*(|\nabla u^*|^2 - 1)dx =$$
$$\int_\Omega \max(0, \lambda^* + c(|\nabla u^*|^2 - 1))(|\nabla u^*|^2 - 1)dx =$$
$$= 0 = c\int_\Omega \max(0, |\nabla u^*|^2 - 1)^2 dx$$

The last expression is as a penalization of the constraint. Moreover, $\lambda_\gamma = c\int_\Omega \max(0, |\nabla u^*|^2 - 1)dx$ is an approximation of the Lagrange multipliers in some sense.

If instead of fixed $2c > 0$ we choose a sequence of penalty parameters $\{\gamma\}$ then we obtain the augmented Lagrangian method for solving of the constrained minimization problem: we solve the sequence of unconstrained minimization problem with the objective functional

$$J_\gamma(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx +$$
$$+ \frac{\gamma}{2}\int_\Omega \max(0, (|\nabla v|^2 - 1))^2 dx.$$

to be minimized over the space $H_0^1(\Omega)$. The objective functional is proper over $H_0^1(\Omega)$, strictly convex and lower semicontinuous, consequently there exists unique minimizer $u_\gamma \in H_0^1(\Omega)$. The necessary and sufficient optimality condition is the following variational problem:

Problem 3 *Find $u_\gamma \in H_0^1(\Omega)$ such that*

$$(\nabla u_\gamma, \nabla v) + (\lambda_\gamma, \nabla u_\gamma \cdot \nabla v) = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega),$$
$$\lambda_\gamma = 2\gamma \max(0, |\nabla u_\gamma|^2 - 1)).$$

We have the following result (we omit the proof here):

Theorem 4 *Assume that there exist Lagrange multiplier $\lambda \in L^\infty(\Omega)$ for the problem (3). For $\gamma \to \infty$ the solutions $(u_\gamma, \lambda_\gamma)$ of problem (3) converges to the solution (u, λ) of problem (3) in the sense that $u_\gamma \to u$ in $H_0^1(\Omega)$ and $\lambda_\gamma \rightharpoonup \lambda$ in $L^2(\Omega)$.*

Although the solution of the original problem is $H^2(\Omega) \cap H_0^1(\Omega)$ its approximation u_γ may not have that higher regularity. Then with $|\nabla u_\gamma| \in L^2(\Omega)$ we cannot obtain Newton differentiability of $2\gamma \max(0, |\nabla u_\gamma|^2 - 1))$ because of the chain rule [5] which requires continuous differentiability of the inner operator. In order to get smoother approximation we perturb problem (3). For the functional

$$J_{\gamma,\varepsilon}(v) = \frac{\varepsilon}{2}(\Delta v)^2 + J_\gamma(v)$$

its minimization problem over the space $H^2(\Omega) \cap H_0^1(\Omega)$ has a unique solution $u_{\gamma,\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$ by the same arguments mentioned above.

Thus we come up with the following modified problem :

Problem 5 *Find $u_{\gamma,\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$ such that*

$$\varepsilon(\Delta u_{\gamma,\varepsilon}, \Delta v) + (\nabla u_{\gamma,\varepsilon}, \nabla v) + (\lambda_{\gamma,\varepsilon}, \nabla u_{\gamma,\varepsilon} \cdot \nabla v) =$$
$$= \langle f, v\rangle \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega),$$
$$\lambda_{\gamma,\varepsilon} = 2\gamma \max(0, |\nabla u_{\gamma,\varepsilon}|^2 - 1)).$$

We have results on the convergence of $u_{\gamma,\varepsilon}$ to u_γ in $H_0^1(\Omega)$ as $\varepsilon \to 0$ which we don't bring here due to the scope of the paper.

Finally, we bring the results on the semismoothness of the Newton method for the problem (5)

Theorem 6 *For fixed $\gamma \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^+$ the semismooth Newton method applied to the problem (5) has superlinear convergence of $u_{\gamma,\varepsilon}^{(k)}$ to $u_{\gamma,\varepsilon}$ in $H^2 \cap H_0^1(\Omega)$ provided that for the initialization $u_\varepsilon^{(0)}$ the norm value $\|u_{\gamma,\varepsilon}^{(0)} - u_{\gamma,\varepsilon}\|_{H^2(\Omega)\cap H_0^1(\Omega)}$ is sufficiently small.*

For the semismooth Newton method the linearization of the nonlinear operator equation $F(u) = 0$ has the form

$$DF(u^{(k)})\delta u = -F(u^{(k)}),$$

where DF is the Newton derivative of F, δu is the update for u. The use of the method leads to the following variational equation at the Newton iteration:

$$\varepsilon \int_\Omega \Delta u_{\gamma,\varepsilon}^{(k+1)} \Delta v + \int_\Omega a^{(k)} \nabla u_{\gamma,\varepsilon}^{(k+1)} \nabla v = \int_\Omega g^{(k)} \nabla u_{\gamma,\varepsilon}^{(k)} \nabla v + \int_\Omega f v, \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega) \tag{6}$$

where

$$a^{(k)} = (1 + 2\gamma \chi_{\mathscr{A}}^{(k)} \cdot (|\nabla u_{\gamma,\varepsilon}^{(k)}|^2 - 1))\mathbf{I} + 4\gamma \chi_{\mathscr{A}}^{(k)} \nabla u_{\gamma,\varepsilon}^{(k)} \otimes \nabla u_\varepsilon^{(k)},$$

with

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$g^{(k)} = 4\gamma \chi_{\mathscr{A}}^{(k)} |\nabla u_{\gamma,\varepsilon}^{(k)}|^2.$$

Here the characteristic function

$$\chi_{\mathscr{A}}^{(k)}(x) = \begin{cases} 1 & \text{if } |\nabla u_{\gamma,\varepsilon}^{(k)}(x)| \geq 1 \\ 0 & \text{if } |\nabla u_{\gamma,\varepsilon}^{(k)}(x)| < 1 \end{cases}$$

describes the active and inactive sets. For the elastoplastic torsion problem the region where the characteristic function for the final solution u vanishes can be interpreted as the elasticity region and the rest of the domain is then the plasticity region where $|\nabla u| = 1$.

The Newton iterations are performed until we get the same active set in two consecutive Newton steps since the equation is uniquely determined by it.

1.3 Algorithm

The whole procedure consists of two nested loops. The outer loop increases the continuation parameter γ. For the initialization we can set $\gamma_0 = 0$, then the corresponding initialization is the solution for the unconstrained problem. In the inner loop Newton iterations for the current value of γ are performed.

Algorithm 1 Semismooth Newton Method

$\gamma := \gamma_0$, choose $u_{\gamma,\varepsilon}^{(0)}$, choose $\varepsilon > 0$
$u^{(c)} = u_{\gamma,\varepsilon}^{(0)}$
while not converged **do**
    $k = 0$
    Set $\mathscr{A}_{\gamma,0} = \left\{x \in \Omega : |\nabla u^{(c)}|^2 > 1\right\}$
    while not converged **do**
        $k = k + 1$
        solve (6) for $u_{\gamma,\varepsilon}^{(k+1)}$
        $\mathscr{A}_{\gamma,k+1} = \left\{x \in \Omega : |\nabla u_{\gamma,\varepsilon}^{(k+1)}|^2 > 1\right\}$
        if $\mathscr{A}_{\gamma,k+1} = \mathscr{A}_{\gamma,k}$ **then**
            STOP
        end if
    end while
    $u^{(c)} = u_{\gamma,\varepsilon}^{(k)}$
    increase γ
end while

2. Implementation using COMSOL Multiphysics

Testing of our algorithm we carried out using GUI of COMSOL Multiphysics 4.1. We were able to show semismoothness of Newton method for the problem working in the function space $H^2(\Omega) \cap H_0^1(\Omega)$ and therefore it's important to test the algorithm with C^1-conforming finite elements method for solving (6). COMSOL Multiphysics 4.1 allows using of Argyris shape functions for it.

In order to implement Newton iterations we used the time-discrete solver in COMSOL Multiphysics. We turned the Newton iterations into a time dependant problem discretized in time with the time step size 1. Thus, the equation (6) represents a discretization in time, $u_{\gamma,\varepsilon}^{(k+1)}$ in the equation corresponds to the current time solution and $u_{\gamma,\varepsilon}^{(k)}$ - the preceding time solution. The reason for using it was that the operator 'prev' corresponding to the previous iteration solution helped us to transfer the gradient data from iteration to iteration.

We describe the main features used in order to implement the continuation method in algorithm (1) by using of COMSOL desktop environment. In the text below '$\rightarrow$' means the choose of subnode or from the corresponding content menu, subsequent menu, panel window, and '=' means the choose from the list.

- We begin modeling with *Equation based modeling→ Weak Form PDE→ Time dependant*. It is necessary to remove the node *Time-dependant solver* and replace it with *Time-discrete solver*.

- In PDE node we set *Discretization→Argyris Quintic*. In *Weak form PDE* node we set the equation form as in (6). To enter $u_{\gamma,\varepsilon}^{(k)}$ in the equation line we use operator prev: e.g. 'prev(u,1)', and for differentiation 'd(prev(u,1),x)'.

- In COMSOL Multiphysics 4.1 *PDE→ Dirichlet Boundary Condition* is not suitable for using of Argyris finite elements on domains with curved boundary due to that the system will then be overdetermined. Instead we use *PDE→Pointwise Constraint* and set the corresponding discretization to '*Lagrange Quadratic*'.

- By *Global Definitions→Variables* we give the data on the exact solution which is known for the elastoplastic torsion problem with $f = d > 0$ constant on the circular domain and its derivatives.

- By *Model→ Definitions→ Model Couplings→ Integration* we set an integration operator with the integration order equal to 10. This operator we use for defining of error norms in function space: *Model→Definitions→Probes→Global Variable Probes*. The values of the defined norms will then be calculated after each iteration.

- In *Study Settings* for the *Time Discrete Solver* we set '*Times = range (0,1,100)*'. Thus the maximal number of Newton iterations is limited to 100 for the case of non-convergence in our test examples.

- The stopping condition defined by the coincidence of active sets as in algorithm (1) we replace with the convergence in $H^2(\Omega)\cap H_0^1(\Omega)$-seminorm

$$\|u\| = \left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{1}{2}}$$

  with tolerance parameter 10^{-10} using the feature *Time Discrete Solver→Stop Condition*.

- For the availability of the preceding solution it's necessary to add a subnode *Solver→Store Solution*.

- MUMPS direct solver was used and the damping was set to constant 1.

- For experiments with the increase of the step size of the continuation parameter γ we first give its initial value, handle initialization of the solution by zero using the *Dependent Variables* node. Then we run the study. For the next values of the the continuation parameter first we need switch on the continuation in the following way : in the settings for the node *Dependent Variable* we set *Initial Values* with '*Method = Solution*' and '*Solution = Solver 1*' (the name of the corresponding solver). The increase of the continuation parameter and running of the study can be done simply 'by hand' or the loop can be automatized by adding in *Job Configuration* node the feature *Parametric*. In the *Settings* panel for the new node we enter the name of the continuation parameter and its range. Then we add a node *Parametric→Solver* setting it with '*Sequence=All*'. It was also possible to use LiveLink for Matlab to fully automatize the outer loop.

3. Numerical results

3.1 Example 1

In this example we consider the elastoplastic torsion problem. A detailed description of the problem can be found in [7], we consider here its general mathematical formulation i.e. the case of infinitely long cylinder and Ω is its cross-section. We consider an example the exact solution for which is known ([2], p.122):

$$\min_{v\in K}\left[\frac{1}{2}\int_\Omega |\nabla v|^2 dx - d\int_\Omega v(x)dx\right]$$
$$K = \{\in H_0^1(\Omega): \quad |\nabla v| \le 1 \text{ a.e. in } \Omega\}$$

hereby, $d > 0$ constant and

$$\Omega = \{x\in\mathbb{R}^2 | \quad x = (x_1,x_2), \quad x_1^2 + x_2^2 < 1\};$$

the exact solution is given by :
if $d \le 2$

$$u(x) = \frac{d}{4}(1-r^2),$$

and if $d \ge 2$

$$u(x) = \begin{cases} 1-r & \text{if } \frac{2}{d} \le r \le 1 \\ -d\frac{r^2}{4} + 1 - \frac{1}{d} & \text{if } 0 \le r \le \frac{2}{d} \end{cases}$$

here $r = \sqrt{x_1^2 + x_2^2}$.

First, we experiment without the smoothing term $\varepsilon \int_\Omega \Delta u_{\gamma,\varepsilon}^{(k+1)} \Delta v$. In Table 1 we bring the detailed convergence results for a triangulation mesh with 1902 triangles and 8873 degrees of freedom (DOF). The continuation method is initialized by zero; the stopping condition for the Newton iterations is $|\|u^{(k+1)}\| - \|u^{(k)}\|| < 10^{-10}$, where $\|u\| = (\int_\Omega |\Delta u|^2)^{\frac{1}{2}}$.

For the loops with $\gamma = 1, 10, 100, 1000$ Newton iterations converge in about six iterations, but for $\gamma = 10^4$ 44 Newton iterations were needed (Table 1). But as the constraint violation term $\int_\Omega \max(0, |\nabla u_h|^2 - 1)^2 dx$ is sufficiently small already with $\gamma = 1000$, in this case there is no necessity to increase γ further. The Figure 2 shows that the free boundary, interface between active and inactive sets of the solution with $\gamma = 1000$ corresponds to that of the exact solution.

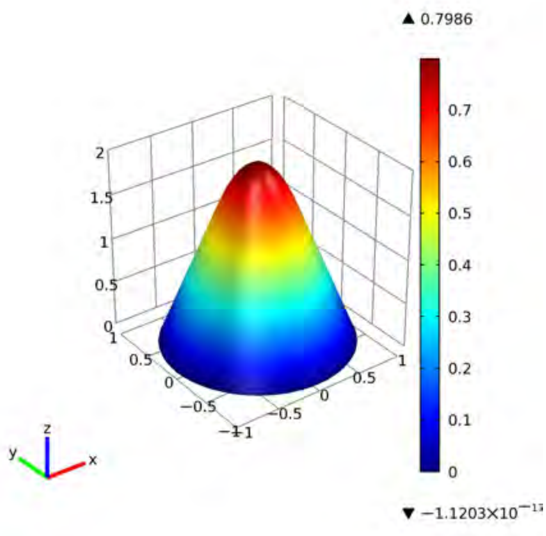

Figure 1: The solution $u_{\varepsilon,\gamma}$ with $\varepsilon = 0$, $\gamma = 1000$

To check the convergence rate of Newton iterations first we increase the number of time-discrete levels up to maximal number of iterations and add the same number of nodes *Solver → Other → Store Solution*. Thus the solutions at the Newton iterations will be available at the postprocessing stage for the calculation of the norm value

$$\|u^{(k)} - u^{(M)}\| = \left(\int_\Omega |\nabla (u^{(k)} - u^{(M)})|^2 \right)^{\frac{1}{2}},$$

where $u^{(M)}$ is the last iteration for the current value of γ. In Figure 3 we present the data on convergence of Newton iterations on a logarithmic scale of $H^2(\Omega) \cap H_0^1(\Omega)$-seminorm with

Table 1: Convergence results for the triangulation mesh with 1902 triangles and 8873 DOF (u_h obtained solution); here CVM is the short notation for $\int_\Omega \max(0, |\nabla u_h|^2 - 1)^2$

γ	iter.	$\lvert u_h - u_h^* \rvert_{H_0^1(\Omega)}$	CVM
0	0	1.6993	-
	1	1.5496	24.2440
1	1	0.7613	3.9270
	2	0.3757	0.7450
	3	0.2620	0.3305
	4	0.2511	0.3005
	5	0.2510	0.3002
	6	0.2510	0.3002
10	1	0.0717	0.0215
	2	0.0384	0.0058
	3	0.0373	0.0054
	4	0.0373	0.0054
	5	0.0373	0.0054
100	1	0.011400874	1.07E-04
	2	0.013605346	6.48E-05
	3	0.013775854	6.45E-05
	4	0.01377592	6.45E-05
	5	0.013775917	6.45E-05
	6	0.013775917	6.45E-05
1000	1	0.016495221	3.53E-06
	2	0.017610584	2.26E-06
	3	0.015743017	1.38E-06
	4	0.015711185	1.31E-06
	5	0.015678803	1.30E-06
	6	0.015677152	1.30E-06
	7	0.015677152	1.30E-06
	8	0.015677152	1.30E-06
1E-4	.	.	.
	43	0.0156	2.9687E-08
	44	0.0156	2.9687E-08

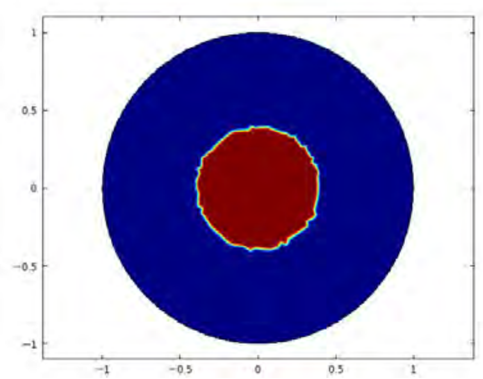

Figure 2: Active (blue) and inactive (red) sets obtained by postprocessing for the solution with $f = 5$

$\varepsilon = 0.0001$. As it can be seen from the figure the convergence rate is superlinear.

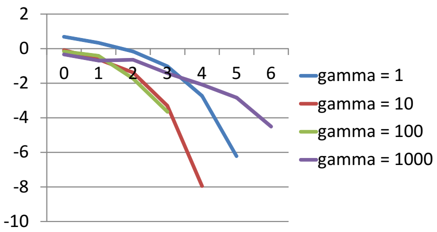

Figure 3: Superlinear convergence of Newton iterations (on a logarithmic scale) with $\varepsilon = 0.0001$; mesh with 1902 triangles

For the semismooth Newton method developed in infinite dimensional space framework the convergence can be expected even for the rapid increase of γ. For instance, we can obtain the solution above without continuation method (Figure 4) simply initializing by zero without deteriorating the convergence. If in order to get the convergence it was necessary to damp Newton iterations or come back to the smaller value of γ and increase it again with smaller step size then this means that semismoothness of Newton method is destroyed by discretization. We tested Algorithm 1 with different finite element implementation in COMSOL Multiphysics and found out that among them Lagrange quadratic and Argyris finite elements demonstrate robust behaviour in respect of refining the mesh.

In Table 2 we present the results for two meshes without continuation method with $\gamma = 1000$. In these tests no damping was done and the stopping condition was convergence in $H_0^1(\Omega)$ seminorm. In case of non-convergence of Newton iterations we force to stop after 100 iterations. We note that for the implementation with Lagrange polynomials we used the feature *PDE $\rightarrow$ Dirichlet Boundary condition* and with Hermite and Argyris polynomials the feature *PDE $\rightarrow$ Pointwise Constraint* with *Lagrange Quadratic* shape functions for the discretization of the boundary condition.

In Table 3 we bring results for tested meshes using continuation with final $\gamma = 1000$. We see that total number of iterations do not increase with the refinement, thus the algorithm demonstrates mesh-independent behaviour. Based on this data the approximate rate of convergence for using of Argyris finite elements for the problem can be obtained (last column). Here we used the maximum element size in the meshes for the calculation of the rate of convergence:

$$\text{convergence rate} = \log_{\frac{h_{1,\max}}{h_{2,\max}}} \frac{\text{error}_1}{\text{error}_2}$$

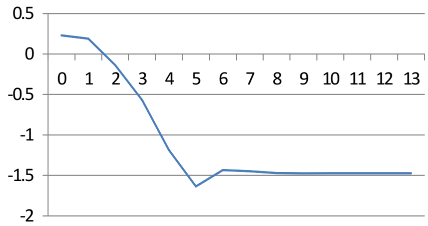

Figure 4: Log scale of the error norm vs iteration number with $\gamma = 1000$; convergence in 13 iterations without continuation initializing by zero

Table 2: Tests with various finite elements implementation in COMSOL Multiphysics 4.1

polynomial	$\lvert u_h - u_h^* \rvert_{H_0^1(\Omega)}$ 938 triangles,	6530 triangles,
Lagrange linear	0.0732 (55 iter.)	0.0525, (100 iter.)
Lagrange quadratic	0.0068 (16 iter.)	0.0012 (15 iter.)
Lagrange cubic	0.0025 (100 iter.)	6.40E-04 (100 iter.)
Lagrange quartic	0.0017 (13 iter.)	5.02E-04 (100 iter.)
Lagrange quintic	0.0014 (20 iter.)	5.02E-04 (100 iter.)
Hermite cubic	0.2896 (100 iter.)	0.1891 (100 iter.)
Hermite quartic	0.0022 (100 iter.)	5.52E-04 (100 iter.)
Hermite quintic	0.0015 (16 iter.)	4.70E-04 (100 iter.)
Argyris quintic	0.0286 (13 iter.)	0.0068 (11 iter.)

3.2 Example 2

In this example we want to show the effect of $\varepsilon \int_\Omega \Delta u_{\gamma,\varepsilon}^{(k+1)} \Delta v$ in the equation on the solution. We choose rectangular domain $\Omega = \{x \in \mathbb{R}^2 | \quad x = (x_1, x_2), \quad -1 \le x_1 \le 1, -1 \le x_2 \le 1\}$ and

$$f(x,y) = \begin{cases} 10\cos(2((y-1)^2 + x^2 - 1)) \\ \text{if } x^2 + (y-1)^2 < 1 \\ 0 \quad \text{elsewhere} \end{cases}$$

First we try with mesh which has coarser structure near the corners of the domain and with $\varepsilon = 0$. Although we obtained convergence of Newton iterations and stopped outer loop with $\gamma = 1000$ (Table 4) Figure 5 illustrates

Table 3: Tests with various meshes; last column: convergence in $H_0^1(\Omega)$-seminorm

# of triangles	# of DOF	# of iter.	$\|u_h - u_h^*\|$	conv. rate
546	2631	13	0.0336	
936	4428	13	0.0286	0.7
1902	8873	11	0.0157	1.7
6530	29951	11	0.0067	1.4
24924	113270	12	0.0026	1.4

Table 4: Test results for the Example 2; here CVM is the short notation for $\int_\Omega \max(0, |\nabla u_h|^2 - 1)^2$

	268 triangles 1352 DOF $\varepsilon = 0$	268 triangles 1352 DOF $\varepsilon = 10^{-4}$	520 triangles 2156 DOF $\varepsilon = 0$
# of iter.	30	34	38
CVM	3.10E-07	2.62E-07	3.25E-07

that higher value of the constraint violation is present on the corner.

Running of the algorithm with the same mesh but with $\varepsilon = 10^{-4}$ smoothed out the higher absolute values of gradient (Figure 6). The improvement was also achieved with $\varepsilon = 0$ and refining the mesh near the corner.

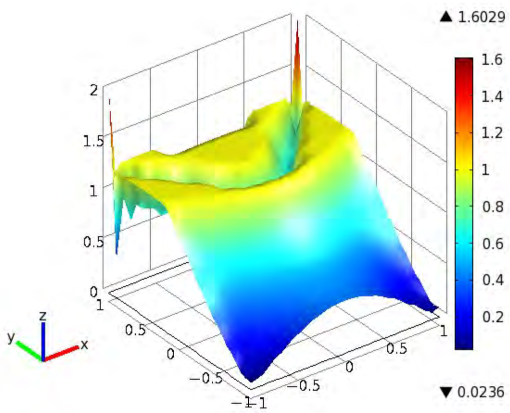

Figure 5: The result with $\varepsilon = 0$, mesh with 268 triangles

4. Conclusions

In this paper we presented the continuation algorithm for the gradient constrained minimization problem and test results obtained by using of COMSOL Multiphysics 4.1. The numerical results of the algorithm implementation using

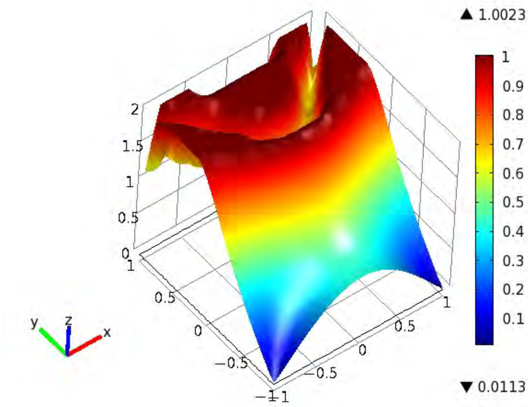

Figure 6: The result with $\varepsilon = 0.0001$, mesh with 268 triangles

Argyris shape functions show mesh independent behaviour of the method. Test results confirm the theoretical part of our research work.

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